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ABSTRACT

The primary objective of this study was to provide an experimental model of children's representations of addition and subtraction concepts viewed as constructed schemes. How children with different counting schemes differ in their addition and subtraction concepts and how the types of problems children solve correlate with the addition and subtraction concepts were specifically explored. The 3-week study was conducted as a teaching experiment, with children's behavior observed and their mental processes probed in interviews, and in teaching episodes. Eight children in grades 1 and 2 were selected to reflect possible variations in counting, addition, and subtraction schemes. Four representations of addition concepts and six representations of subtraction concepts were found, with one or more specific schemes identified with each representation. The schemes were classified by developmental levels. Children who constructed higher level schemes also solved all kinds of addition and subtraction problems which involved larger numbers. Children's uses of their schemes reflected awareness of the difficulty of a problem and basic understanding of adding and subtracting. (MNS)

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# Representations of Children's Addition and Subtraction Concepts

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## Abstract

The counting unit types, counting perceptual, motor, verbal, and abstract unit items have been identified with children's schemes for adding and subtracting. The observations on eight first- and second-graders involved in a three-week teaching experiment, provided four representations of children's addition concepts, and six representations of their subtraction concepts. One or more specific schemes have been identified with each representation. The schemes have been classified according to developmental levels, with the children capable of using the more advanced unit types, constructing the higher level schemes. Children who constructed higher level schemes also solved all kinds of addition and subtraction problems, which involved larger numbers. Children's uses of their schemes reflected the awareness of the difficulty of a problem, and their basic understanding of adding and subtracting.

Representations of Children's Addition and Subtraction Concepts

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Despite the long history of educational reforms the education community is still unclear as to what we mean by educating children in schools, even when we restrict our concern to a subject area like mathematics. What are the givens in the school setting? Is it the child or a predetermined scope and sequence of a mathematics curriculum? Let us quickly cast our mind into a classroom where a teacher is helping a child, Hendry , a six-year old, to solve "6 + 3". The teacher is frustrated because despite all her efforts and encouragement little Hendry can simply not utter "6-7,8,9" and give "9" as the answer. Hendry has to use blocks, count out six blocks and then count three more blocks in a separate location. Finally he makes a single heap of all the counted blocks and sequentially touches the blocks while uttering "1,2,3 ... 9". The teacher's intention is to teach Hendry to count on from "6" but the latter can only count perceptual items (blocks, fingers), and has yet to construct unit items from number word utterances.

The above episode supports the view that teachers need to be fully aware of the child's processes for representing addition and subtraction of numbers, as well as the unit items the child is capable of creating while counting. It is this knowledge that would enable the teacher to provide appropriate opportunities that could help to bring out the most prominent mathematical knowledge in the child - an ideal goal which Kireenko (1959) has called "pedagogical optimism" (p. 19).

## Purpose

The primary objective of the study reported in this paper is to provide an experimental model of children's representations of addition and subtraction concepts viewed as constructed schemes. The study will also investigate the possible indicators of developmental itineraries of adding and subtracting schemes that are identified in the experimental model. In particular, we will investigate how children who possess different counting schemes differ in their addition concepts and subtraction concepts, and how the types of problems children solve correlate with their addition concepts and subtraction concepts.

## Rationale

On every occasion that the child attempts to solve or correctly solves an addition or a subtraction problem, the child reveals something about her knowledge of the arithmetical operation in particular and her knowledge of number in general. It is not that the child creates or constructs a piece of knowledge according to the rules and structure of the mathematics she is starting to learn; rather the child re-presents the mathematical operation (an internalized activity in the Piagetian sense) in whatever way she can. This global re-presentation is a reflection of the way the child, at that moment in her development, organizes her experience.

The Piagetian principle is taken that the child necessarily has to construct her own representations of reality – even if, at some later point of development, the child's subjective reality does become or is expected to be compatible with the reality of the social group in which she operates. Steffe, von Glasersfeld, Richards, & Cobb (1983) have demonstrated that the concept of unit, that forms the basis of all numerical operations and arithmetic skills is developed out of the child's own constructions which pass through a sequence of stages. The counting types of Steffe et al. (1983) provide a firm foundation from which to explore the steps children take when constructing addition and subtraction concepts.

Addition and subtraction constitute a major part of the elementary school mathematics curriculum, especially in the first and second grades. The importance of the need for teachers to provide appropriate opportunities for children to construct and use addition and subtraction concepts cannot be overemphasized. This need has been a recurrent theme in recommendations made by researchers investigating children's concepts or skills for solving addition and subtraction problems. Brownell (1928) addressed this need in the following recommendation:

> teachers must keep fully informed concerning the stages of development of the pupils by means of continuous study ... of the procedures and processes which the pupils employ in dealing with numbers (p. 143).

More than 20 years after Brownell's recommendation, Ilg and Ames (1951) found the need to make a similar recommendation,

> Not only is it important to know more about each individual child's developmental rate in regard

> to mathematics, but also we should know more about each individual child's particular processes, number systems, ... and devices which he uses in arriving at answers to arithmetic problems (p. 26).

Most first- or second-grade teachers will confirm that they have observed significant individual differences regarding the processes and the strategies their pupils use to solve addition and subtraction problems, despite the common instruction the teachers provide. Appropriate instruction, which takes the children's differences into consideration requires knowledge of the addition and subtraction concepts children do construct and how they construct them. Researchers (Brownell, 1928; Carpenter, 1983b; Carpenter & Moser, 1981, 1982; Davydov & Andronov, 1981; Groen & Resnick, 1977; Houlihan & Ginsburg, 1981; Ilg & Ames, 1951; Siegler & Robinson, 1982; Steffe, Thompson, & Richards, 1982; Steffe, von Glasersfeld, Richards, & Cobb, 1983; Suppes & Groen, 1967; Woods, Resnick, & Groen, 1975) have attempted to map the processes children use to solve addition and subtraction problems for more than half a century. Despite the well-documented strategies that children use to solve these problems, Carpenter and Moser (1981) have rightly pointed out that there is "a great deal that is yet unknown about how addition and subtraction concepts and skills are acquired" (p. 62).

Providing models that are based on minute analyses of observations of children's constructive activities in the context of solving problems is one viable approach to elucidating children's acquisition of adding and subtracting schemes. No two children may be exactly the same with respect to their intellectual development. While there always may be some significant differences in the construction of adding and subtracting schemes by any two children, a teacher's knowledge of the ways and means of the acquisition of these schemes may greatly facilitate her ability to foster the adaptations made by particular children.

The teacher with the goal of fostering adaptation should be generating hypotheses about her children's actions, interpreting their behavior, and evaluating these hypotheses to provide evidence and support for her future decisions. The teacher should have the specific objective of applying, immediately, her knowledge and the interpretation of her observations to assist the children she is teaching. The significance of the study is enhanced by the benefit a teacher can derive from the models provided in her diagnosis and direction of children's construction of addition and subtraction concepts. The inclusion of children's counting unit types in the representation of addition and subtraction concepts in the study opens up another dimension for research into how children come to acquire and construct these concepts.

## THEORECTICAL FRAMEWORK

### Teaching and Learning

#### The Child

Psychologists and educators have long recognized the child as the center of interest (Knight, 1930, p. 3) in education. Some will go further and claim the child as also the most important component of education (Gattegno, 1970, p. ii). Knight (1930a) rightly points out that it is inadequate to base teaching solely on the "interest and felt needs" of the child. But he favors strict adherence to the organized "curriculum laid down before the child enters school" (p. 6). Hence, the curriculum becomes the most important component and not the child as suggested by Gattegno (1970). Focusing on the child should result in organizing the curriculum not beforehand, but according to the mental powers of the child. Gattegno (1970) calls these powers the "functionings of children" (p. 7). To have a functioning is to have the "know-how" to function in a particular way. Some important examples of these functionings for learning mathematics are the power of extraction, the power to make transformations, the power to make abstractions, and the power of imagery.

Knowledge

Focusing on the mental powers of the child leads to the primary concern of synchronizing teaching with children's mathematical thinking. Thus, to be successful, an approach to teaching should lead to the generation of knowledge by the students. The most important question is, "how does the child come to have particular knowledge"? The answer we give to this question depends on our view of the nature of knowledge.

In the decade from the 1970 to 1980 the hopes the "New Math" had kindled for the teaching and learning of mathematics disappeared. The mathematics education community which, in the sixties, appeared to know "where it was going" gradually realized that children were not performing any better in mathematics than the children of previous decades. In fact, many children left the study of mathematics as soon as possible (Hurd, 1982). Complaceny had to give way to a period of "groping for a clearer focus and sense of direction" (Hill, 1983, p. 1). The failure of the New Math movement was ironically due partly to the apparent success claimed by the mathematics educators involved in identifying the type of knowledge children needed to acquire. As von Glasersfeld put it:

> Educators were concerned with getting knowledge into the heads of their students, and educational researchers were concerned with finding better ways of doing it. There was then, little if any uncertainty as to what the knowledge was that students should acquire, and there was no doubt at all that, in one way or another, knowledge could be transferred from a teacher to a student. The only question was, which might be the best way to implement that transfer (p. 42).

This approach to teaching is based on the assumption that the teacher possesses knowledge which she imparts to the student. Gattegno (1970) characterizes this approach as the "subordination of learning to teaching" (p. 5), and illustrates it as in Figure 1 (p. 3). In this scenario, knowledge is supposed to exist independently of the student and can be passively transferred to him or her by the teacher. The student is supposed to need only memory in order to receive knowledge (Gattegno, 1970, pp. 3-4).

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Insert Figure 1 about here

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von Glasersfeld (1983) provides a sketchy but adequate historical review of the traditional conception of knowledge. He concluded that a dilema arises when we accept the traditional conception of knowledge "that requires a match or correspondence between (our) cognitive structures and what these structures are supposed to represent" (p. 48), because in this scenario, "truth" becomes the perfect match, that is, a flawless representation. He argues that, since we are logically incapable of having a "God's eye view" (Putnam, 1981) of the real world and its presumed representation, there is no way out of the dilema (p. 48). However, he suggests we can resolve our experiential problems by adopting the kind of knowledge that fits human observations. From this perspective, the world we live in is always and necessarily the world as we conceptualize it. But we still cannot make "facts" as we like. For as von Glasersfeld put it:

> They are viable facts as long as they do not clash with experience, as long as they remain tenable in the sense that they continue to do what we expect them to do (p. 51).

If we take this latter view of knowledge then our approach to teaching and learning should differ from that illustrated in Figure 1. For in this (second) scenario, the student will have to organize what the teacher says to fit his or her own knowledge. Knowledge, then, is constructed by the student as opposed to being transferred ready-made by the teacher. This approach to teaching and learning is consistent with what Gattegno (1970) calls "the subordination of teaching to learning" (p. 14), which he illustrates as in Figure 2 (p. 14).

Insert Figure 2 about here

It should be emphasized, however, that the communication between the student and the teacher is crucial even though the child has to construct its own knowledge. We shall see the need for this communication later when we consider the teacher's role in the child's learning situation.

Theories of Learning

Since the turn of the century psychologists have propounded a number of theories and explicated how these theories could optimize the learning of mathematics (especially arithmetic). The connectionism of Thorndike (1922) attracted many disciples. Knight (1930b) based his treatment of teaching methods on Thorndike's connectionism and provided extensive treatment of drill on the basic facts. The Gestalt psychology developed by Kohler (1929) and Wertheimer (1923 / 1938) endeavored to explicate more complex learning in problem solving and understanding of mathematical stuctures than the connectionism of Thorndike (1922) could explain. Katona (1940 / 1967) extended the principles of Gestalt psychology to explicate the distinction between "senseless" (rote) and "meaningful" (understanding) learning. An important issue in learning theories is the transfer of knowledge gained in learning one task to another. Gagne (1962, 1970) initiated the cumulative learning theory in which he explicates how complex skills can be analyzed into ordered subskills, or learning hierarchies.

Piaget (1964) has not provided any explicit theory of learning that can be applied directly for instruction. His views are that development of knowledge explains learning rather than the widely held opinion that development is a sum of discrete learning experiences (p. 8). Piaget (1970b) has identified three categories and meanings of experience that contribute to cognitive development. First, there is simple exercise in which the child acts on objects without extracting any knowledge from them. But the exercise may, if exploratory in nature, provide new exogenous information as well as consolidate the child's activity. Second, there is physical experience which enables the child to extract information from the objects themselves using simple (empirical) abstraction. Here the child discovers new properties of the objects while it disregards others (e.g. discover weight while disregarding color). Third, there is logico-mathematical experience which is an important component in cognitive development and allows the child to discover new deductive instruments. This experience enables the child to derive knowledge based on his actions on objects rather than from their physical properties. Piaget (1970b) emphasizes that knowledge acquired through experience has two poles: "acquisition derived from the objects and constructive activities of the subject" (p. 721). Piaget (1964) cautions that while it is possible to obtain learning through physical experience by external reinforcement, learning that involves the construction of a logical structure cannot be obtained by external reinforcement (p. 16). Piaget (1964) points out that such learning may be possible only if the subject (learner) already possesses the necessary and supportive simpler, more elementary logical-mathematical structures required for the structure to be taught (p.16).

The Teacher's Role

Perhaps the greatest attraction of behaviorism is in the possibility of providing specific and direct guidelines for instruction. The teacher's role is well defined in instruction based on behavioristic principles. However, the state of the art of mathematics teaching indicates that clarity of purpose and specified sequence of instructional steps do not in themselves guarantee success in learning by students.

We take the view that knowledge is not passed on to the passive student by the teacher (Gattegno, 1970; Piaget, 1970a). The student generates knowledge through his or her actions (transformations) carried out on objects (Piaget, 1970a) or through interaction with the teacher (Gattegno, 1970) (see Figure 2). Thus the teacher should not consider herself as a repository and a transmitter of knowledge (Vergnaud, 1983) to the student. The teacher's role is to create the enviroment that is most congenial for the child to interact with her (teacher). The most important task of the teacher is to select appropriate activities that the child can carry out. The teacher has then to determine "where the child is" from the child's behavior as the latter performs the activites. The teacher's understanding of the child's knowledge will be compared with the teacher's goals for the child and adjustments made in the latter. As the child is working the teacher carefully intervenes, providing the guidance and support that is necessary to enable the child to make progress. Also, the teacher's intervention could be in the form of asking new questions to enable the child to reflect, if possible, on her experience of doing mathematics.

To understand the child's actions and responses, the teacher must formulate hypotheses about the child's capabilities and the possible progress she (child) can achieve. This calls for testing the hypotheses which ultimately leads to the generation of further tasks and hypotheses . The teacher, in essence, will take on the role of an hypothesis formulator and tester with the specific objective of using her knowledge about how a particular child thinks and acts. The teacher's goal is to lead the child in the acquisition of a knowledge or of a method to solve a task. But the teacher must allow the child to generate her own conceptions or methods.

The role of the teacher suggested above is consistent with the constructivist approach to teaching (Cobb & Steffe, 1983; von Glasersfeld, 1983). The principles of constructivism may therefore have little meaning and application for the teacher who believes, first, that she needs a textbook that explicitly directs all her mathematics teaching; second, that children should be taught in large groups with a minimum of child initiated communication; third, that the child's own constructed methods for solving arithmetical tasks are unimportant and should be ignored and; fourth, that children need to be drilled to acquire adult methods and procedures. The "constructivist teacher" is required to be creative and sensitive to each child's mathematical knowledge. This situation places a great deal of responsibility on the teacher as well as presenting an enormous problem when the teacher comes to grips with the slow pace of progress in some children. But the teacher should be encouraged by the fact that her success will not depend on how much of the adult concepts and methods the child is able to master. Rather, her goal should be to bring out the most prominent "mathematical" knowledge in the child. This can only be achieved through the intelligent use of the powers of the mind of all concerned, both the teacher and the student (Gattegno, 1970, p. ii), thus making the communication between the teacher and the child the most crucial aspect of teaching.

The Figurative and Operative Aspects of Thought

Piaget (1970a) distinguishes two apects of thinking that are different, but complement each other. He calls these the figurative and operative aspects. In Piaget's (1970a) view, the essential aspect of thought is its operative and not its figurative aspect (p. 15). But to obtain a complete picture of children's mental development , we must consider the figurative aspect in addition to the operative aspect (cf. Steffe, 1983). To Piaget (1970a), for a child to know an object or some "reality", that child must act on the object and transform it in order to understand how a certain state is brought about (p.15). The operative aspects of thought are the set of actions and operations (internalized actions) of the child that attempt to transform reality (Piaget, 1970b, p. 716). Thus Piaget

(1970b) points out that "operative" is a broader term than "operational", as the latter is only related to the operators (p. 716). On the other hand, the figurative aspects are the activities that attempt to imitate reality: perception, imitation (including graphic imitation or drawing), and mental imagery (cf. Bruner's (1966) enactive and iconic representations).

Empirical and Reflective Abstractions

The type of knowledge that a child derives from an object depends on the sources of the child's abstractions. Piaget (1970a) provides two sources from which the child can abstract. First, there is the object itself, and second there are the actions carried out on the object. Piaget (1970a) calls the knowledge or abstraction derived from the object itself "empirical" knowledge or "simple" (empirical) abstraction. The knowledge or abstraction drawn from the coordination of actions, and not from the object, is called logical - mathematical knowledge or "reflective" abstraction, using this term in a double sense. For as Piaget (1970a) explains,

> "Reflective" here has at least two meanings in the psychological field, in addition to the one it has in physics. In its physical sense reflection refers to such phenomenon as the reflection of a beam of light off some surface onto another surface. In a first psychological sense abstraction is the transposition from one hierarchical level to another level of action (for instance, from the level of action to the level of operation). In a second psychological sense reflection refers to the mental process of reflection, that is, at the level of thought a reorganization takes place (pp. 17-18).

The construction of units or unitary items (Steffe et al., 1983; von Glasersfeld, 1981) by the child from things (sensory material) constitutes an instance of pseudo-empirical abstraction (von Glasersfeld, 1981, p. 52). When the child takes these units as material and unites them to construct a number (composite unity) then he or she has made a reflective abstraction. Here the action is a mental act which is reversible and therefore an operation.

## <u>The Counting Types</u>

Steffe et al. (1983) have demonstrated, through minute analyses of first-grade children's counting behavior in the context of solving arithmetiical tasks, that children construct five different types of units.

<u>Counting</u> is defined as "the production of a sequence of number words, such that each number word is accompanied by the production of a unit item" (Steffe et al., 1983, p. 24). Each counting type is based on the type of unit item the child appears to create and is aware of during counting. This view agrees with that of Bridgman (1959) when he pointed out that,

> "the thing we count was not there before we counted it, but we create it as we go along. It is the acts of creation that we count" (p. 105).

The unit items are the objects that are created by the child as she isolates and focuses on certain sensory-motor signals, i.e. visual, auditory, and tactual perception , and also proprioceptive sensation. The five counting types, in order of sophistication, are counting perceptual, figural, motor, verbal, and abstract unit items. As the creation of these unit items constitutes a developmental progression, the child who is a counter of a particular unit item is capable of creating and counting more primitive items. Thus a child is classified as a counter of the most advanced unit items that she can create while counting. Figure 3 (Steffe, et al., 1983, p. 117), schematically represents the hierarchical classification of counters and the types of items that they are aware of and can create during counting.

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Insert Figure 3 about here

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<u>Counting Perceptual Unit Items</u>

Steffe et al. (1983) refer to counting that takes perceptual items as units as <u>counting perceptual unit items</u>. The child who requires the perceptual component and is unable to count unless a collection of perceptual items is actually availabe is called a <u>counter of perceptual unit items</u>. The child's global experience comprises not only the perceptual signals from which the perceptual items are isolated and constituted into countable units, but also the child's motor acts (pointing or nodding), and the vocal production of number words. But the child has no awareness that the number words designate the numerosity of consecutively produced perceptual unit items.

<u>Counting Figural Unit Items</u>

The first step towards independence from perceptual signals in counting is when the child develops ability to abstract figural representations of perceptual items (i.e. visualized images). This becomes necessary when perceptual items presented in a task are hidden from the child's view under a screen. The child's figural re-presentations of the inaccessible perceptual items may be complete or only partial representations. A child is called a <u>counter of figural unit items</u> if the child is able to construct and count figural representations of perceptual items which, though they are presented in the context of a task, are not perceptually available at the moment. The child's counting actions are necessarily restricted to the area of the screen concealing the perceptual items, because the child is counting the hidden perceptual items. the child is unaware of the motor acts (pointing or touching specific locations over the screen) for isolating the visualized images into discrete experiential items.

<u>Counting Motor Unit Items</u>

The next step from counting with figural items is when the child differentiates its motor activity from other components of a counting act (e.g., visual perception, utterance of a number word). The child can therefore execute its motor actions intentionally as well as in absence of perceptual items. The motor act can now be taken by the child as a unitary event that has a beginning and an end. So the motor act becomes a substitute for countable perceptuual items, and the child attends to the motor act as a unit item. A child is said to be a <u>counter of motor unit items</u> whenever the child's countable items are limited to perceptual, figural, and motor units (Steffe, et al., 1982, p. 85). Such a child still requires the actual performance of a motor act, accompanied by the utterance of a number word, in order to create a countable item. The counting activity has yet to be fully internalized by the counter of motor unit items.

Counting Verbal Unit Items

The vocal production of a number word is itself a motor act. But it is a special kind of motor act, because the proprioceptive sensations arising from the beginning and end of the utterance of a number word are less apparent. The child has, therefore, more difficulty in considering its utterances as discrete items. When the child has developed greater self-awareness so that number words can be used as substitutes for items of the previous types, she is called a <u>counter of verbal unit items</u> (Steffe, et al., 1983, p. 120).

Counting Abstract Unit Items

Counters of verbal unit items are not capable of what is generally called "double counting." To do so requires the awareness that the numerical structure designated by a number word is a composite of individual units. Steffe et al. (1983) call "a child a <u>counter of abstract unit items</u>' only when she has acquired the capability of going from a number-word to the conceptual structure which that word designates, i.e., to the intenal construct that constitutes the particular <u>numerosity</u>" (pp. 120-121). For example, to solve "7 + 4", the counter of abstract unit items will simply utter, "seven" then continue with "eight, nine, ten, eleven" while extending four fingers to keep track of his or her counting acts. By uttering, "seven", the child can unite the counting acts, "one, two, three, ... , seven" into a numerical structure (composite unity). This is an example of integration and Steffe et al. (1983) call it tacit integration (p. 69) because the counting acts were implied in the utterance of "seven" but not actually performed.

The counter of abstract unit items itentionally keeps track of its counting acts because they are instantiations of the numerical structure to which "four" refers. Upon arrivig at the number word "eleven", the child can take it to designate the numerical structure that contains 11 individual units. The child who is a counter of abstract unit items does not need to be given any perceptual items in order to create the conceptual structure that constitutes countable items. But such a child can take any sensory-motor unit itself as an abstract unit.

Counting Schemes

Counting is an activity that is repeatable and can therefore be considered as a scheme for, as Piaget (1980) points out, "all action that is repeatable or generalized through application to new objects engenders by this very fact a 'scheme'" (p. 24). But a scheme is not simply activity. von Glasersfeld (1980) describes the complex nature of schemes as follows:

> Schemes as basic sequences of events that consist of parts. An initial part that serves as a trigger or occasion. In schemes of action, this roughly corresponds to what behviorists would call "stimulus", i.e., a sensory-motor pattern. The second part, that follows upon it is an action ("response") or an operation (conceptual or internalized activity) ... The third part ... is what I call the result or sequel of the activity (p. 81).

The child's counting activity in the context of solving addition and subtraction is therefore a scheme. What triggers or presents an occasion for counting depends on the child's previous experiences. For example, the numeral "4" or number word "four" may not itself trigger counting by a counter of perceptual or motor unit items. The child may have to associate the numeral or number word with a collection of items or a figural pattern which then sparks off in the child an intention to count. The second part of the counting scheme is a constructive activity in which the child creates and counts countable items. As pointed out in the last section, the countable items (from the child's view) will be perceptual, figural, motor, verbal, or abstract unit items. The counting scheme for a child will therefore include the countable unitary items that she is capable of creating. The period (length of time interval) during which a child creates her most advanced unit item is referred to as the period of the counting scheme. The counting scheme periods are referred to as the perceptual, figural, motor, verbal, or abstract periods, respectively.

## The Role of a Function Machine

The "function machine" is described in the literature (McKillip & Davis, 1980) as a device that accepts numbers in the form of input numerals and does something to these numbers, giving the result as output numerals. Even though it is possible to construct a function machine that can actually carry out an internal transformation on the input numerals to produce output numerals (cf. the calculator or the microcomputer), the idea has been to use a simple box with openings, say, marked "IN" and "OUT" as illustrated in Figure 4.

Insert Figure 4 about here

The adult is aware that what the machine does to the input numerals is an imagination of the mind but the child would usually "play the game" and believe that the machine actually does something to the input numerals. The function machine therefore provides an ideal device that can be used by the adult to simulate the operations of addition and subtraction. Second, the device can be used to present problematic situations that translate into addition, subtraction, missing-addend, missing-subtrahend, missing-minuend, comparison, and equalizing problems.

The working or operational steps of the function machine can be considered as the analogue of the parts of a scheme. The first step, acceptance of input numerals, can be compared to the first part, the trigger of a scheme of action. Next, the second step, giving a result as an output numeral, can be compared to the third part, the result or sequel of a scheme. The child, presented with an operation performed by the machine, witnesses these two steps. But the child does not "see" how the machine carries out the second step, the analogue of the activity or operation of a scheme.

Let us suppose the input numerals are "5" and "3" and the machine "does something to these numerals" and yields the output numeral "8". We can then ask a child to infer the operation (internalized or hidden action) she believes the machine carried out. If the child is able to respond correctly that the machine performed an "addition", then we can infer that her knowledge of addition is an internalized or mental action of which she is aware. Because, in order to respond correctly, the child had to know either the addition fact "5 + 3 = 8" or be able to mentally add 5 and 3 and compare her result with the output numeral "8". Such a child would have demonstrated an operative (Piagetian sense) concept of addition.

Another child might count her fingers or objects and compare the last number word "eight" uttered in the counting acts with the output numeral. This child may select "addition" or "plus" as the operation carried out by the machine but use different expressions. For example, the child may say, "the machine makes 5 and 3 become 8". Further, if we replace the input numerals with cards showing pictures of animals, say birds on a card, then the latter child might not think in terms of the action on numbers, but physically transform the pictures from two cards onto a third card, the output. This should indicate that the child has not acquired addition as internalized action. To the child addition has to be acted out by manipulating objects or their mental re-presentation, perhaps as a recognizable figural pattern. The child's concpet would still be operative but the actions would be actually carried out. At best this child can imagine and imitate actions but might not be able to transform the action into a complete internalized activity.

Given that a child believes that the function machine carries out an operation or performs some action on the input numerals, the child can be asked to indicate her conception of this operation or action. That is, the words (addition, plus, put together, makes a new number) a child uses to describe what she thinks the machine does can be used to present problems to investigate her concept and representation of addition. For example, the child is shown two numerals on cards, say "7" and "8" which are then given to her to put into the machine. The child is then told that the machine "adds" or "puts together" (or using her own words for describing addition) the two input numerals, and she is to figure out the output numeral. To present a missing-addend problem, the child is not shown one of the input numerals, and either the interviewer puts the numeral into the machine or the child is asked to shut her eyes before given the numeral and guided to put it into the machine. The child is then asked to take out the output numeral and figure out the unknown (missing) input numeral. It is hypothesized that what the child imagines the machine will do constitutes her _representation_.

The child's scheme for adding will reflect her concept of addition. By using the function machine to present addition problems, the child will enact re-presentations of her concept of addition. Since the problem situation involving input numerals into the machine may be different from the contexts in which the child would have previously learned to use her adding scheme(s), the child will have to assimilate the new situation into her existing scheme(s) for adding. Once the child has adapted to this situation, subsequent tasks will provide a context for her to modify her existing adding schemes in solutions. Similarly, the function machine can be used to investigate the child's concept and representation of subtraction.

## Modeling

Models are useful in the detailed analysis of children's mathematical constructions and abstractions in the context of solving arithmetical problems. We use "model" in the sense of cybernetics rather than of replica of a physical object, say, aeroplane. The goal is to hypothesize conceptual structures and systems of transformations that account for the mathematical behavior (overt or inferred) exhibited by the children under observation. A viable way to provide models of children's addition and subtraction concepts is, therefore, to attempt to reconstruct the steps that led the children to whatever conception of these arithmetical operations they might have acquired. The reconstruction is purely hypothetical and is based on our (observers') interpretation of the children's behavior in performing the tasks involving addition and subtraction. Steffe et al. (1983) point out this limitation in modeling the child's conception of number and numerical operations:

Such a reconstruction is necessarily hypothetical,

> because another person's conceptions are, by definition not observable. In this connection it is crucial to remember that conceptual structures (knowledge) are, in our view, not transferable ... even if a child were aware of its own conceptions and could reflect upon and verbalize them, even if it told the observer (teacher, experimenter) what it believes to be, say, its concept of number, that observer could not but interpret that verbal message in terms of his or her own experience (p. xvi).

However, the models that are constructed from interpretation of observations made on some children will remain viable as long as they (models) are not confounded by other experience (or experiment). The models may then be used for predicting or explaining future experience (or child behavior). Nevertheless, from the constructivist's viewpoint, the viability of a model does not exclude its replacement by another. Secondly the viability of a model is not only due to the nature of the model, but also to the characteristic way of conceptualizing the experiences portrayed by the model. Since the constructivist believes that the child's concepts are constructed from its own experience, any model of the child's concepts entails making certain inferences about the child's experience (which is invariably different from the observer's).

One basic activity from which children construct addition and subtraction concepts is counting. The use of counting by young children to solve addition and subtraction problems, from simple numerical combinations to verbal story problems, have been very well documented (Brownell, 1928; Carpenter & Moser, 1982; Carpenter, Hiebert, & Moser, 1981; Davydov & Andronov, 1981; Fuson, 1982; Gelman & Gallistel, 1978; Groen & Parkman, 1972; Groen & Resnick, 1977; Starkey & Gelman, 1982; Steffe, Spikes, & Hirstein, 1976; Steffe, Thompson, & Richards, 1982; Steffe, von Glasersfeld, Richards, & Cobb, 1983; Suppes & Groen, 1967; Thaeler, 1981; Woods, Resnick, & Groen, 1975). Nevertheless, the work of Steffe and his collaborators (1976, 1982, 1983) has provided an alternative and promising theoretical framework to account for the constructions and development of children's addition and subtraction concepts. Other researchers (Carpenter et al., 1981; Carpenter & Moser, 1982; Houlihan & Ginsburg, 1981) who have modeled children's addition and subtraction concepts have significant differences in their theorectical framework. First, their analyses of children's behavior are based not on constructivism but on behavioristic or information processing paradigms (Greeno, 1976). Second, mathematical knowledge is assumed to exist in the environment independent of the child (human organism), and it can be passed on directly by the teacher to the child. Thus Carpenter (1983a) mentions the reduction of "mathematics to a series of component skills that can be taught directly" (p. 104). Steffe et al. (1983) share Piaget's (1970a) views concerning the growth of mathematical knowledge:

> I think that human knowledge is essentially active. To know is to assimilate reality into systems of transformations. To know is to transform reality in order to understand how a certain state is brought about. By virtue of this point of view, I find myself opposed to the view of knowledge as a copy, a passive copy, of reality (p. 15).

What a particular child assimilates as knowledge as a result of her actions or operations, will therefore greatly depend on the child's previous knowledge or experience. The child's experience is necessarily different from that of the observer (adult), and may differ from that of another child. Third, all children who can count to solve arithmetical tasks are considered to be numerical (Starkey & Gelman, 1982). Thus the children's actions are operative (Piagetian sense). But Steffe (1983) has shown that the solution processes of some children, though sophisticated in appearance, are still figurative and involve no operations (interiorized actions) but figural images and re-presentations. Fourth, though these researchers have identified the use of fingers and number word utterances in counting processes, there is no attempt to discriminate between the intentions of the children. As was explained in the last two sections, one child may count fingers as perceptual items, another as motor items, and a third as abstract items. Children using these different conceptions of units have been shown (Steffe et al., 1983) to differ significantly in their understanding of and solution processes for addition and subtraction problems.

## METHOD

The study was conducted as a teaching experiment. This involved observing the children's behavior and probing their mental processes during clinical interviews. There were also teaching episodes during which the interviewer communicated with the child in an attempt to encourage the child improve her counting skills, number word sequences, use of spatial and finger patterns, and to show flexibility in the use of her counting, adding, and subtracting schemes.

## Subjects

The subjects were eight first- and second-grade children in an elementary school in the Clarke County School District in Georgia, which serves both middle and working class communities. The experiment was conducted towards the end of Spring 1983 when the first-grade children had received instruction in both addition and subtraction. The eight children were selected from 17 others after about 20 to 40 minutes interview with each child individually. The children were selected to reflect the possible variations in counting, adding, and subtracting schemes that was evident from the interviews. There was an equal number of males and females as well as an equal number of first- and second-graders.

## Materials

Two types of devices constructed with boxes were used as "function machines" to simulate addition and subtraction operations and to present all types of addition and subtraction problems (see Tables 1 through 6). The first function machine

Insert Tables 1 through 6

about here

has two input holes, marked "IN" and an output hole, marked "OUT" (see Figure 4(a) ). The second function machine has only one input hole and an output hole (see Figure 4(b) ).

Numerals written on cards were used as input and output numerals for the machines. Also pictures of animals on cards and objects in sandwich bags were used as input and output numerals. Blocks of centimeter cubes were available for the children to use to form collections when necessary.

Procedure

The experiment was performed during normal school time and the children were taken into a small room individually to be interviewed and taught. Each session was video recorded and lasted from 20 to 30 minutes. The interviewer worked with each child for 4 to 6 sessions and the video tape for each child was analyzed before the next session. This enabled the interviewer to formulate hypotheses about the child's counting, adding and subtracting schemes and plan appropriate activities and problems for presentation at the next session.

The addition and subtraction problems were orally presented to each child using the function machines. Tables 1 through 6 show the different problems used but the number and types of problems presented to each child depended on his or her performance.

First, the function machine was introduced to each child as a device that accepts input numerals (numerals or pictures on cards and objects in bags). The child was then told the machine will do something to the input numerals and give the result as an output numeral. After this, the child was shown two numerals and requested to put them into the machine and take out the output numeral. The child's task was to determine what she thought the machine did to the input numerals that resulted in the output numeral obtained.

Second, the child's expression for describing the operation was used to present problems involving that operation. The addition operation was used first in one or two sessions, depending on the performance of the child, before the subtraction operation was introduced. More addition and subtraction problems, as well as comparison and equalizing problems were presented in later sessions.

Parts of some sessions were used as teaching episodes. The final session for each child was used to conduct an interview to determine the child's counting type.

## RESULTS

### Addition Concepts

The children's knowledge of addition as an operation were revealed in their responses for describing or characterizing an addition imagined to go on inside the "function machine" (see Figure 5(a)). Table 7 shows the various characterizations of addition by the children. Only three of the eight children used

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Insert Table 7 about here

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the usual words, "add" or "adding" to describe the operation. Two other children described the operation as "put ... together to make ..." (i.e. put 5 and 3 together to make 8). These five children seemed to have abstracted the input numbers five and three from the pictures of birds on cards inserted into the machine. They performed a mental addition (interiorized action) and compared their results with the output numeral "8", which was also abstracted from the pictures of birds on a card drawn from the machine. We hypothesize that these five children had an _operative_ concept of addition. The children succeeded to transform what appeared to be a transfer of pictures on two cards onto a single card into a situation that involved mental addition of numbers. They did not simply attempt to imitate the action they believed the machine carried out.

On the other hand, the remaining three children did not make any numerical transformation. They focused their attention on the pictures of birds as input and output, and attempted to describe the possible action they believed the machine to have carried out. Their descriptions of the operation were "take away ... put them on here" and "makes more numbers". Thus they attempted to imitate or present a mental imagery of what might have taken place inside the machine. They did not attempt to transform the perceived situation. We hypothesize that these three children had a _figurative_ concept of addition.

## Representations of Adding Schemes

Four representations of children's adding schemes that reflect their addition concepts have been identified . The representations, A1, A2, A3, and A4 (see Figures 6 through 9) were observed in the context of children solving simple addition, and missing addend (first or second) problems (see, Tables 1 and 2). A basic scheme for solving an addition problem underlies each of the four representations. Also one or more specific schemes have been identified with each representation.

Insert Figures 6 through 9 about here

Representation A1

The basic scheme underlying this representation is referred to as Counting All. In this representation (see Figure 6) the child counts a collection of items starting from one till she utters the number word for one addend. The child then counts a second collection of items starting from one again till she utters the number word for the second addend. Next, the child might bring all the items in the two collections together to make a single collection as shown in the diagram after the arrow in Figure 6(a), and then counts the items starting from one. Alternatively, the child may leave the two collections where she initially established them, and count all the items starting from one by making an enactive bridge from one collection to the other as shown in the diagram after the arrow in Figure 6(b). The child takes the last number word she utters as the answer. Since the items used to establish the two collections will be recounted they must necessarily be perceptual in order to leave permanent records for the recounting. The following are the two specific schemes that have been identified with this representation.

1. Counting All In Joined Collections. In this scheme (see Figure 6(a)) the child physically combines the items in the two collections to form a single collection, and then proceeds to count all the items, starting from one. For example, asked to solve "2 + 5", and directed to use blocks, Hendry counted two blocks and then five more in a separate location. Next, he took the two blocks and added them to the five others and then counted all the blocks uttering "1,2,3 ... 7" in synchrony with touching the blocks.

2. Counting All In Separated Collections. In this scheme (see Figure 6(b)) the child keeps the two collections in their separate locations, disregards the physical separation, and counts all the items by making an enactive bridge from one collection to the other. For example, Monica solved "6 + 8" by first counting 6 blocks, placed them in a row, and then counted 8 more blocks in another row below the 6 blocks. She then recounted all the blocks without, first, combining them into a single row.

Representation A2

The basic scheme underlying this representation is referred to as Counting From 1. In this representation (see Figure 7) the child starts to utter number words from one, and continues till she utters the number word for one of the addends. The child may or may not keep track of her counting acts up to this point. But the child then makes a separation in her counting activity, and continues to utter number words forwward while keeping track of the counting acts by extending fingers or touching objects. The child stops uttering number words when the number (from the child's point of view) of items used as records equals the second (other) addend, or when the child utters the number word for the sum (if solving a missing-addend problem). The child takes the last number word uttered as the answer, or the number of recorded items as the answer (if solving a missing-addend problem). The following are the three specific schemes identified with this scheme.

1. <u>Counting Perceptual Unit Items From 1</u>. In order to use this scheme correctly, the child should sequentially touch objects or her fingers in synchrony with uttering number words from one. The child would stop counting appropriately if she made a separation after uttering the number word for the first (selected) addend and recognized a spatial pattern that corresponded to the second addend. The two children who attempted to use this scheme both failed to make separations or use spatial patterns. For example, Hendry (6yr. 9mo), a counter of perceptual unit items, sequentially touched all his fingers till he ran out in synchrony with uttering "1,2,3 ... 10", in an attempt to solve "3 + 4". Similarly, Monica (8yr. 5mo.), a counter of motor unit items, realized she needed more than her 10 fingers to solve "7 + 8". But when the interviewer suggested she could use some of his fingers, Monica simply counted all the 20 fingers, her 10 fingers as well as those of the interviewer's. Monica nodded in agreement when the interviewer asked if she was sure of the answer.

2. <u>Counting Motor Unit Items From 1</u>. In this scheme the child counts her motor acts beginning to utter number words from 1 till she utters the number word for the first (selected)

addend. The child then continues to utter the succeeding number words in synchrony with producing motor acts equal to the second addend. The difficulty with this scheme is the need to keep track of the counting acts for the second addend. Children use finger patterns to enable them to overcome this difficulty. It is important for the child to make a visual or physical separation between items representing the two addends. For example, Paris (8yr. 4mo.), a counter of abstract unit items, failed to make a separation when solving "7 + 8". He extended 18 fingers while uttering "1,2,3 ... 7-8,9,10 ... 18". Even though Paris paused after 7 before resuming counting, he appeared to have counted 8 more fingers after he had used all his 10 fingers. This accounted for why Paris stopped counting after uttering "18" and seeing a pattern of his 8 extended fingers. Paris had no difficulty in using this scheme to solve "4 + 10". There was a natural separation after counting his 10 extended fingers so he easily extended four more and stopped.

3. Counting Verbal Unit Items From 1. In this scheme the child utters the number word sequence from 1 till she utters the number word for the first (selected) addend. The child then pauses momentarily before continuing to utter succeeding number words. The child may rely on the rythmic pattern in the utterances of the number words corresponding to the second addend in order to know when to stop. Shani (7yr. 1mo.), a counter of verbal unit items, recalled "11" after few seconds in answer to "3 + 8". Asked to pretend to explain how she solved the problem to her friend, Shani replied,

> "I would say, 1,2,3 ... 8. Because that is the hard one. 9,10,11"(emphasizing "11").

Shani deliberately decided first to utter the number words from one through eight, because she foresaw it was going to be difficult for her to keep track if she began with the number words from one to three. The raising of her voice to emphasize "11" and the fact that she stopped uttering number words indicate her awareness of having counted three more. Shani confirmed her inability to keep track of longer number word utterances by refusing to attempt "7 + 8". She complained, "I can't do 7 and 8 because that is a long one. I can do 7 and 2".

Representation A3.

The basic scheme underlying this representation is referred to as Counting On. In this representation (see Figure 8) the child utters the number word for one given addend, and then continues to utter the succeeding forward number words. The child keeps track of her number word utterances mentally or by extending fingers or by using objects as records. To solve an addition problem, the child stops uttering number words when the number of items used as records is the same as the other given addend. The child then takes the last number word uttered as the answer. To solve a missing-addend problem, the child stops uttering number words when she utters the number word for the given sum. The child then takes the number of items recorded as the answer. The following are the five specific schemes that have been identified with this representation.

1. <u>Counting On-Using Perceptual Unit Items</u>. When the child intentionally counts on, using perceptual items, especially objects other than her fingers, then we say the child is employing the Counting On-Using Perceptual Unit Items scheme. For example, to solve "30 + 13", Valerie (7yr. 3mo.), a counter of abstract unit items, spontaneously asked for blocks to count. After establishing two separate collections of 30 and 13 blocks, Valerie touched the collection of 30 blocks once and said, "30". She then continued to count the other 13 blocks by uttering "31,32,33 ... 43" in synchrony with touching the blocks. When she was done, she said, " that's 43". Valerie's action indicated that she intended to count on from 30. This observation is supported by the fact that she had counted on, using motor unit items in the preceding task, "7 + 8". But she needed to count from one before she could construct the numerical structure, 30. After counting 30 blocks she knew there were 30 individual blocks so she made an integration of her previous counting acts when she repeated "30". We hypothesize that Valerie could not, however, make a tacit integration of 30 counting acts as she succeeded to make for eight counting acts to solve "7 + 8".

2. <u>Counting On-Using Motor Unit Items</u>. When a child relies on her motor acts to count on, we say that the child is using the Counting On-Using Motor Unit Items scheme. Cullen (6yr. 9mo.), a counter of abstract unit items, solved the missing addend task, "34 + ___ = 44" by sequentially extending her fingers in synchrony with uttering, "35,36,37 ... 44". She then said, "10" while looking at her open two hands. Cullen's intention was to

count her motor acts of putting up fingers. When she was done she did not know how many number words she had uttered, until she saw her 10 fingers all extended. We infer that Cullen counted on using motor unit items rather than verbal unit items. This does not exclude the fact that she could have constructed abstract units from her motor acts.

3. Counting On-Using Verbal Unit Items. In order to count on using verbal unit items without using finger movements to keep track of how many number words have been uttered, the child resorts to the use of rythmic (temporal) patterns. Jeff (8yr. 5mo.), a counter of abstract unit items, used a temporal pattern of two number words followed by a string of five number words to solve "7 + 8". He uttered "8-9,10-11,12, 13,14,15". The break in his utterances after "10" enabled him to mentally keep track of the next five number words. On other occasions when children were observed to count on addends greater than seven, their number word utterances were accompanied by finger movements. In some cases the children used their fingers subtly, say, by pressing them on their thighs.

4. Counting On-Using Abstract Unit Items. The counter of abstract unit items is capable of taking any sensory-motor unit as an abstract unit (Steffe et al., 1983, p. 67). When such a child clearly shows that she counted on to solve an addition problem but used none of the observable sensory-motor units, then she is classified as Counting On-Using Abstract Unit Items. For example, Jeff used abstract units to count on from 8 to solve "8 + ___ = 11". He sat silently for about 10 seconds before saying the answer was "3". The following portion

of his protocol (I: interviewer, J: Jeff) shows how he solved the problem:

I: Can you tell me how you found out?
J: I had 8 and you had 3, and together makes 11.
I: How did you know I had 3?
J: I counted them.
I: What did you say when you counted?
J: 8-9,10,11.

Jeff had earlier in the same interview recalled immediately that "8 + 3" is "11". If he had related this addition fact to the missing-addend problem then he would have used it to solve the task. But when Jeff was asked how he got "3", he replied, "I counted them". We infer that Jeff counted as he claimed and constructed three abstract units from the internalized counting acts, "9,10,11".

5. Adding On By Tens And Ones. This scheme involves adding, first tens to one addend followed by adding ones. This scheme was used to solve a missing-addend problem by John. He gave "34" as the answer to "10 + ___ = 44" and explained as follows:

"I added 3 tens to 10. That will be 40. And I added 4 to the 3 which gives you 34. And you need 10 to make 44".

John's explanation indicated that he added 3 tens to 10 to obtain 40. He immediately realized that he needed 4 more to make 44. So rather than continuing to obtain 44 after saying "40", John went on to complete his goal of finding the missing addend. He added 4 to 30 to obtain 34. John's use of "3" rather than "3 tens" or "30" reflects his strong reliance on place value ideas (see the Recalling Sums Using Place Value scheme below).

Representation A4

The basic scheme underlying this representation of additon is referred to as Recalling Sums. In this representation (see Figure 9) the child searches for and finds an addition fact from memory that involves both or one addend and modifies it. As soon as two numbers in the recalled additon fact fit the given numbers in the problem, the third number is taken as the answer. The following are the four specific schemes that have been identified with this representation.

1. Recalling Sums By Guessing. This scheme involves the child recalling an addition fact immediately (about 2 seconds) after the problem was presented. With regard to the representation shown in Figure 9 the child's action follows only the arrows 1 and 4 without following arrows 2 and 3. That is, the child obtains the required sum without any intermediate partial sum. The child who is not "guessing" but has meaningfully habituated (Brownell, 1928) the addition fact should be able to explain her answer by using one of the already discussed schemes. Children who used this scheme and obtained wrong answers sometimes admitted that they guessed. Children also usually followed a wrongly guessed answer with other guesses.

2. Recalling Sums Using Doubles. This scheme involves the child recalling an addition fact that is the double of one of the given addends. The child then increases or decreases the "partial sum" by a number that is the difference between the given addends to obtain the required sum. To solve a missing-addend problem, the child increases or decreases the doubled addend by the difference between the given sum and the partial

sum to obtain the misssing addend. For example, John (8yr. 2mo.), a counter of abstract unit items, solved "15 + ___ = 31" mentally and explained his solution as "Fifteen plus 15 is 30, and you need one more to make 31". John constructed the missing addend from 15 and one. He obtained 15 as part of the missing addend ( the invterviewer's input numeral which John did not see) by recalling the doubles fact "15 + 15 = 30". He then realized the partial sum, 30 had to be increased by 1 to obtain the given sum, 31. So he also increased 15 by 1 to obtain 16 as the missing addend.

3. Recalling Sums To A Decade. In this scheme the child recalls an addition fact that involves adding a number to the larger addend to yield the next decade. The child then increases the decade by the difference between the other addend and the number added to the larger addend. The answer is the final sum obtained. To solve a missing-addend problem, the child increases or decreases the given addend to obtain the next or the preceding decade respectively. The child then adds the increment to or subtracts the decrement from the difference between the given sum and the decade to obtain the missing addend. For example, to sovle "15 + ___ = 31", John sat for some time and said, "You put in 16". His explained his answer as follows:

> "You already have 15 and you add 5 makes 20, and 10 more makes 30. That's 15 and 1 more is 16".

We infer that John intended to add numbers to 15 till he got the decade nearest 31, that is, 30. He therefore added five to 15 to get the decade, 20 and then added 10 more to get the decade, 30. John kept track of the numbers he added to 15, and realized that he had used 15. He needed one more to make 31 so he added it to 15 to get 16, since his goal was to find how many he should add to 15 to get 31.

4. Recalling Sums Using Place Value. This scheme involves the child recalling the sum for the numbers in the tens and ones places separately and coordinating the two sums to form the appropriate number. If the sum of the numbers in the ones place is greater or equal to ten, the sum obtained for the numbers in the tens place is increased by one (or one ten). For example, John explained his answer to "13 + 15" as, "the 3 and 5 gives you 8, and 1 plus 1 is 2. So it must be "28". We infer that John was aware that he was adding tens when he said, "1 plus 1 is 2". So he mentally converted the two tens into 20 and coordinated the eight ones with it to obtain 28.

## Subtraction Concepts

The children's knowledge of subtraction as an operation was revealed in their responses for describing or characterizing a subtraction imagined to go on inside the function machine (see Figure 5(b)). Table 8 shows the observed children's characterizations of subtraction. Hendry and Shani described the

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Insert Table 8 about here

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operation as "make less" and "changed it from six to four " respectively. Monica's description was, "it took two away" (pointing with her finger to show the two empty spots on the output card). These three children's descriptions suggested that they attempted to describe the physical action that they imagined the machine to perform on the inputs (pictures of animals on cards). There was no effort to transform the situation into a numerical operation that could be carried out mentally. However, Hendry and Shani later selected "take away" from the suggestion, "did the machine add, subtract, or take away"? We infer that these three children had a _figurative_ concept of subtraction.

Cullen, Jeff, Paris, and Valerie described the subtraction they imagined to go on in the machine as "take away". John was the only child to use the "mathematical" term "subtract" to describe the operation. These five children's description involved the mental action of taking away or subtracting two from six. Their responses indicated that they transformed the collections of animals into numerical structures and operated on the numbers mentally. We infer that the five children had an _operative_ concept of subtraction.

_Representations of Subtracting Schemes_

Six representations of children's subtracting schemes that reflect their subtraction concepts have been identified. The representations, S1, S2, S3, S4, S5, and S6 (see Figures 10 through 15) were observed in the context of children solving

Insert Figures 10 through 15

about here

direct subtraction, missing-subtrahend, missing-minuend, comparison-more, comparison-less, equalizing-add, and equalizing-take away problems (see Tables 3 through 6). A basic scheme for solving a subtraction problem underlies each of the six representations. But one or more specific schemes have been indentified with each representation which reflect the integration of counting unit items in the children's schemes for subtracting.

Insert Tables 3 through 6
about here

Representation S1

The basic scheme underlying this representation of subtraction is referred to as Separating. In this representation (see Figure 10) the child counts perceptual items to construct a collection equal to the larger given number in a problem. The child then separates and counts items equal to the smaller given number to establish a new collection. Finally, the child counts the remaining items of the first collection and takes the last number word she utters as the answer. Only perceptual items which can leave permanent records for the second and third counting activities, could be used with this scheme. The specific scheme for this representation is, therefore, referred to as Separating. The following illustrates how Shahi, a counter of verbal unit items, used this scheme to solve "9 - 5".

> Shani sequentially took 9 blocks from a box and placed them in her hand, while uttering "1,2,3 ... 9". She arranged the blocks in a row, and sequentially separated 5 blocks in synchrony with uttering

"1,2,3,4,5". Next she counted the remaining 4 blocks by subvocalizing number words and said, "4" (with emphasis).

Shani transformed the task into "how many blocks will be left from nine if she took away five"? This is indicated by her subvocalizing the counting acts for the blocks left. She was only interested in the last number word so she emphasized it to signify that she was done.

Representation S2

The basic scheme underlying this representation of subtraction is referred to as Adding All. In this representation (see Figure 11) the child counts perceptual items to construct two collections equal to the given numbers in the problem. The child then recounts all the items in the two collections by making an enactive bridge from one collection to the other or by first, joining the collections to form a new single collection. The specific scheme identified with this representation is referred to as Adding All. This was similar to the Counting All In Separated Collections identified as an adding scheme, but the analogue of the Counting All In Joined Collections scheme was not identified in the study. The following is part of a protocol which s how Monica used this scheme to solve "___ - 2 = 4", presented with blocks in sandwich bags.

I: So how many did I put in? (The interviewer had put in the minuend without showing it to Monica).
M: 3 (Guesses).
I: How are you going to find out?
M: Count on my fingers.

I: Can you do so?
M: (Simultaneously extends two fingers on her right hand and four fingers on her left hand). 1, 2, 3 . . . 6, in synchrony with folding down her extended fingers.

After guessing "3", Monica reorganized her thinking, and perhaps recalled how the interviewer had previously guided her to solve a similar task "___ - 3 = 5". She established the two collections for "2" and "4" simultaneously and then counted all extended fingers. Monica did not juxtapose the extended fingers on both hands before counting.

Representation S3

The basic scheme underlying this representation of subtraction is referred to as Trial And Error. In this representation (see Figure 12) the child constructs a collection equal to the larger number. The child then guesses the answer to the problem and separates items equal to this number, and counts the remaining items. If the last number word uttered is not equal to the smaller given number, the child increases or decreases the guessed number and repeats the separating action. The child continues to modify the guessed number until the remaining items after separating some items, equal the smaller number. The child takes the last number word uttered in the final counting activity as the answer. The specific scheme for this representation is referred to as the Trial And Error Separating. The following illustrates how Cullen used this scheme to solve "13 - ____ = 9".

> Cullen counted 13 blocks and made them into a heap. She then separated 6 blocks, counted the remaining blocks, and found there were 7 left rather than 9. She recombined the blocks, separated 5 blocks, and counted the remaining 9 blocks. She answered, "5".

When Cullen found that there were seven blocks left, she realized that the separated blocks were two more than she wanted. So she decreased seven by two and separated five blocks. Had Cullen known that five was the answer, she would have uttered it immediately. But her scheme required that she checked to see that nine blocks were left. Hence her need to perform the second separation of blocks and count the remaining blocks.

Representation S4

The basic scheme underlying this representation is referred to as Counting Up. This scheme is similar to the Counting On scheme for addition. In the representation (see Figure 13) for this scheme, the child utters the number word for the smaller or any given number, and then continues to utter succeeding forward number words. The child keeps track of her number word utterances mentally or by using fingers or objects as records. The child might count up to the other given number, and take the number of counting acts recorded as the answer. The child might also count up as many times as the second given number, and take the last number word as the answer. The four specific schemes identified with this representation were as follows:

1. Counting Up-Using Abstract Unit Items.
2. Counting Up-Using Verbal Unit Items.
3. Counting Up-Using Motor Unit Items.
4. Counting Up-Using Perceptual Unit Items.

The following illustrates the Counting Up-Using Motor Unit Items scheme to solve "____ - 7 = 7".

> Valerie said, "It's like adding two sevens up. 7-8, 9, 10 . . . 14", in synchrony with extending 7 fingers. "Fourteen".

Valerie intended to start at seven, then extend and count seven fingers. She also knew that the last number word will be the answer, so she repeated it as soon as she was done.

Representation S5

The basic scheme underlying this representation is referred to as Counting Down. In this representation (see Figure 14), the child might begin by uttering number words backward from the larger given number and proceeds counting to the smaller given number, keeping records of the number of counting acts. The child takes the number of counting acts recorded as the answer, and the scheme is referred to as Counting Down To. The child might also begin by uttering number words backward and stop after recording as many counting acts as the smaller given number. The last number word is taken by the child as the answer, and the scheme is referred to as Counting Down With. The six specific schemes identified were as follows:

1. Counting Down To-Using Verbal Unit Items.
2. Counting Down To-Using Motor Unit Items.
3. Counting Down To-Using Perceptual Unit Items.
4. Counting Down With-Using Verbal Unit Items.
5. Counting Down With-Using Motor Unit Items.
6. Counting Down With-Using Perceptual Unit Items.

To solve "13 - ____ = 9", Jeff uttered "13-12,11-10-9", and answered, "3 - 4", without any finger movements. We infer that Jeff used the Counting Down To-Using Verbal Unit Items scheme. Our hypothesis is supported by the fact that Jeff paused after 11, and after 10. These pauses enabled him to keep track of his number word utterances.

The following illustrates how Paris used the Counting Down With-Using Perceptual Unit Items to solve "18 - 7".

> Paris uttered "18-17,16-15-13", in synchrony with folding all his left fingers. He stopped, counted the folded fingers, and continued "12-10", and folded two more fingers. Paris then answered, "10".

Paris intended to count backward seven times, by creating seven perceptual items. He did not maintain a mental count of his folded fingers so he interrupted the counting activity to find out. The pauses in his counting acts was the effect of his poor backward number word sequence. Our hypothesis is supported by Paris' ommission of "14" and "11". Also, Paris did not count the motor acts of folding fingers, because he did not look for finger patterns.

Representation S6

The basic scheme underlying this representation is referred to as Recalling Number Facts. In this representation (see Figure 15) the child recalls number facts from memory until she recalls an addition or a subtraction fact that involves two of the given numbers in the problem. The child then takes the third number as the answer. The following are the five specific schemes of the representation.

1. Recalling Result By Guessing. This scheme is similar to the Recalling Sum By Guessing scheme for adding. But in this case, the recalled number fact can be an addition or a subtraction. The child who use's this scheme, with understanding, should be able to explain how his answer could be obtained by using one of the other subtracting schemes or an adding scheme. For example, Paris solved "9 - 6" by saying, "3" immedately after the problem was

presented. He explained his answer by extending nine fingers, folded down six fingers, held the three remaining fingers, and said, "three will be left".

2. Recalling Addition Fact. In this scheme the child recalls an addition fact that involves the two given numbers and relates it to the problem. The child takes the third number, which was not given, as the answer. This scheme follows the path given by the arrows 1 and 6 in Figure 15. For example, to solve "how many more is 5 than 3", Valerie said, "2, because 2 and 3 is 5". We infer that Valerie recalled the addition fact "2 + 3 = 5", and related it to the given numbers in the problem. She then realized that 3 and 5 were the given numbers, so she gave "2" as the answer.

3. Recalling Subtraction Fact. This scheme is similar to the Recalling Addition Fact scheme, but in this case, the child recalls and relates a subtraction fact to the given problem. For example, Jeff recalled the subtraction fact, "5 - 2 = 3" and related it to the problem "____ - 2 = 3". His answer "5" was the number that was not given in the problem.

4. Recalling Number Facts By Trial and Error. In this scheme the child recalls successively, at least two Addition or Subtraction Facts that comes to his mind until the recalled number fact involves the two given numbers. The child then relates the number fact to the problem and takes the third number as the answer. The child uses this scheme when he does not know how to modify his first recalled number fact to make it involve the two given numbers. For example, to solve "24 - 12", John recalled the addition fact "14 + 12 = 26". He realized it did not relate to the problem, that is the sum was not 24. So he recalled "12 + 12

= 24", which related to the problem, and took "12" as the answer. John's explanation for making a second recall was that, "I used 14 and 12, and got 26, but that didn't work, so I tried 12 plus 12 and that worked". We infer that John was not aware that if he added 12 and 12 he will get 24 until he recalled the number fact "12 + 12 = 24". Also, John did not realize that decreasing either 14 or 26 by two will give the correct result. John's explanation indicated that he was merely trying another number fact to see if "that will work".

5. Recalling Derived Facts. In this scheme the child recalls an addition or a subtraction fact that involves at most one of the given numbers. The child then modifies (increases or decreases) the numbers until he obtains a number fact that involves the two given numbers. The child takes the third number, not given, as the answer. For example to solve "14 - ____ = 7", John said, "7 and 6 is 13 and 1 more is 14", and answered, "7". John's first recalled addition fact, "7 + 6 = 13" involved only one given number. So he knew he was not done. John, therefore, increased the sum by 1 and mentally increased 6 also by 1. This led him to recall the number fact, "7 + 7 = 14", so John took "7" as the answer.

## DISCUSSION

### Adding Schemes

Piaget (1970a) identified four fundamental characteristics of an operation:

> First ... an operation is an action that can be internalized; that is, it can be carried out in thought as well as executed materially. Second, it is a reversible action ... Third ... it always supposes some conservation, some invariant ... fourth ... (an) operation is related to a system of operations, or to a total structure (pp. 21-22).

Simply put, an operation is an internalized and reversible action that is invariant and is embedded in a system governed by rules. Addition satisfies these four characteristics with subtraction as its reversal and with the system of the whole numbers as the underpinning structure. The adding schemes that have been identified and which involve counting, that is, Counting All, Counting From 1, and Counting On reflect the material execution rather than the internalized action of addition. In other words, even though these schemes are based on mental re-presentations of abstractions from previous experiences, they are actions that have not been completely internalized. The activities of uttering number words and creating unit items (perceptual, motor, verbal) form an integral part of the child's means for executing the action of adding. On the other hand, the non-vocalized schemes, that is, Recalling Sums and Counting On-Using Abstract Unit Items are completely internalized actions.

Steffe et al. (1983) characterized children's counting schemes for adding as <u>extensions</u>. These extensions were identified mainly in the context of children solving tasks involving partially or totally hidden collections. The adding schemes that have been identified in this study with children solving tasks presented mainly in the context of the function machine are consistent with the extensions identified by Steffe et al. (1983). This supports the view that the addding schemes and especially the counting unit types (counting perceptual, motor, verbal, or abstract unit items) are normal constructions by children as they acquire numerical concepts. That is, the counting types are not constrained to the singular context of having children count partially and totally hidden collections. These same schemes can be observed in totally different contexts.

Developmental Levels in Adding Schemes

Counting All. This scheme was identified as the lowest level of children's representation of addition. This was the only scheme Hendry (6yr. 9mo.) and Monica (8yr. mo.) , counters of perceptual and motor unit items respectively, used correctly to solve addition problems. Paris (8yr. 4mo.), a counter of abstract unit items used this scheme, but it was prompted by a suggestion from the interviewer to use blocks after failing several attempts to count on to solve "30 + 13" (probably due to his faulty number word sequence beyond "30"). Valerie (7yr. 3mo.), a counter of abstract unit items also used some parts of this scheme, but because she counted on from 30 after establishing two collections for "30" and "13", her scheme was classified as Counting On-Using Perceptual Unit Items.

The Counting All scheme requires three counting sequences (see Figure 6). But even the Counting All In Separated Collections scheme cannot be considered as an example of Steffe et al.'s (1983) simple extension. Because the child's intention is to count a single collection just as in the Counting All In Joined Collections scheme. The only advance is that the child realizes before counting that the physical separation between the two collections can be ignored. The activity part of this scheme is included in Houlihan and Ginsburg's (1981) Counting From 1 using concrete aids. Carpenter(1983b) and Hiebert et al. (1982) also classified their Counting All strategy as the lowest strategy children use to model addition.

Counting From 1. This scheme was identified as an advance over the Counting All scheme for two reasons. First, the Counting From 1 scheme included the ability to count verbal unit items which the children who were limited to the use of the Counting All scheme lacked. Second, the children who used the Counting From 1 scheme solved a wider range of problems than those who used only the Counting All scheme.

Shani (7yr. 1mo.), a counter of verbal unit items, used the Counting From 1 scheme to solve nearly all her addition tasks, including missing-addend tasks. Paris was the only counter of abstract unit items who used this scheme. Monica, a counter of motor unit items, used the Counting Perceptual Unit Items From 1 scheme. But her solutions were incorrect because of her failure to make separations in the counting acts in order to keep track of the addends, especially the second. Monica usually used her motor acts of finger extensions to establish two collections to represent the addends on separate hands. She then recounted the fingers as perceptual items. Monica's scheme was identified as Counting All, because Monica could not count her motor acts to establish a single collection that included both addends. Hendry also failed to use the Counting Perceptual Items From 1 scheme correctly, because, like Monica, he also failed to make separations in the counting activity.

Some reseachers do not distinguish between the counting activity ("response") in the Counting From 1 and the Counting All schemes. Carpenter (1983b) and Hiebert et al. (1982) identified only Counting All, and Houlihan and Ginsburg (1981) identified only Counting From 1 using or not using concrete aids. The latter included "counting in which the child claims to have just counted numbers" (p. 99). We also observed this situation but we did not identify this as Counting Abstract Unit Items From 1 scheme (see Table 9). First, counting abstract unit items was only attributed to children classified as counters of abstract unit items and therefore able to consturct a numerical structure (cf. Steffe et al., 1983, p. 68). Second, counters of abstract unit items when counting mentally are likely to count on than count from 1. Counters of verbal and motor unit items did perform mental addition but their explanations indicated that they re-presented to themselves spatial or figural patterns or imagined counting acts which they then mentally monitored and counted (cf. the example given below of how Shani solved "5 + ___ = 7"). Our claim is consistent with Houlihan and Ginsburg's (1981) report that "the child claims to use some kind of mental picture such as dots or lines" (p. 99). Thus our definition of Counting Perceptual, Motor, or Verbal Unit Items From 1 includes the counting of the re-presentations of counting activity using these unit items. However, we hypothesize that when a counter of abstract unit items re-present a sensory-motor item to herself, the child constitutes, the item as an abstract unit.

The Counting Verbal or Motor Unit Items From 1 schemes are classified by Steffe et al. as <u>intuitive extension</u> and the Counting Perceptual Unit Items From 1 scheme is called <u>simple extension</u>.

<u>Counting On</u>. This scheme was identified as the second highest level of the observed children's representation of addition. Cullen (6yr. 9mo.) and Jeff (8yr. 5mo.) as well as Paris and Valerie, all counters of abstract unit items used mainly the Counting On scheme. Cullen, Jeff, and Valerie used four out of the five Counting On schemes (see Table 9). Paris gave no indication that he could count on, using abstract unit items (that is, count on mentally and respond correctly to the problem). He always had to verbalize his thoughts after sitting quietly for a long time before he produced correct responses.

The only occasion Shani could be classified as using the Counting On scheme was when she solved "5 + ___ = 7". But the evidence is suspect because Shani initially guessed "13", "9", "11" and "10" as answers in that order and then said, "2 more". Her explanation w s, "My mind said 6 and 7 so I said 2 more". Shani's actions indicated that she first attempted to add mentally 5 more to 7. Her incorrect responses indicated her inability to count on correctly beyond 7 by 5 more. The most plausible interpretation is that Shani did not count on but re-presented the counting acts, "1,2,3 ... 7" to herself but only vocalized the part of the counting acts, "6,7" to show how she figured out the answer. This interpretation is consistent with the Counting From 1 scheme she used frequently. Monica and Hendry also gave no indication that they could use the Counting On scheme.

Houlihan and Ginsburg (1981) identified two strategies of the Counting On. They included the use of fingers in the Counting On-Using Concrete Aid strategy. We distinguish between the use of fingers as perceptual and motor items, and identified these two uses with separate schemes. Carpenter (1983b) and Hiebert et al. (1982) identified two strategies, Counting On From First (smaller) Number and Counting On From Larger Number, which focus on the addends in the problems rather than the qualitative differences in the items used for counting on. We did not always make the first addend smaller and the children usually counted on from the larger addend. One child, Cullen even asked in her first interview, "Can I start with 45", when solving "16 + 45". This indicated some children might count on consistently from the first (smaller) addend because they are not sure it is permissible to reverse the order of the addends. Cullen's question indicated that she was aware that the results would be the same when she reversed the addends, but she did not know that she would be allowed to do so.

Steffe et al. (1983) restricted the scheme, numerical extension , to the counting on scheme when the child constructs a numerical structure during the process of counting. All the children in the study who counted on also used the scheme to solve missing-addend problems that involved an addend at least greater than 10 (see Table 10) thus these children could construct numerical extension schemes.

Recalling Sums. This scheme was identified as the highest level of representation of addition by the children in the study. This is consistent with the findings of Carpenter (1983b). Steffe et al. (1983) did not elaborate on children's use of this scheme but only counters of abstract unit items were identified with the use of this scheme. For example, Christopher used what we call Recalling Sums Using Doubles and Recalling Sums To A Decade to solve "7 + 5" and "6 + ___ = 10". (pp. 106-107). Table 9 shows that one child, John (8yr. 2mo.), a counter of abstract unit items, answered nearly all problems presented to him by Recalling Sums. His answers clearly indicated that he could recall many addition facts from memory. His only use of the Counting On scheme was when he added on by tens and ones to solve "10 + ____ = 44". But he showed he could ount on in a problem situation when he solved how many were in a known hidden collection and an unspecified number of visible items. This problem was similar to that used by Steffe et al. (1983), and it was used in the determination of his counting unit type.

Recalling sums using doubles or to a decade is contingent upon a child's ability to recall immediately the addition fact involving the doubles or the decade number. For example, John's use of the doubles fact to solve "15 + ___ = 31", depended on his immediate recall of "15 + 15 = 30". Cullen, Jeff, and Valerie also used the Recalling Sums scheme. But they all used only the Recalling Sums By Guessing scheme. That is, most of their recalls were made within two seconds after being presented with a problem. In addition none of their explanations for recalled responses included the use of the doubles fact nor the addition fact to a decade. They usually explained mental additions by counting on. For example, Cullen immediately recalled "2" as the answer to "5 + ___ = 7", but explaining her solution, she said, "I counted in my mind, 5-6,7". It was possible Cullen knew that "5 + 2" is "7" but her explanation was consistent with her dominant scheme, Counting On, for solving addition.

Paris and Shani also used only the Recalling Sums By Guessing scheme but they gave incorrect responses. However, Paris and Shani gave some correct recalls which were clearly (from later explanations) the results of mental adding of re-presented counting acts. Monica and Hendry also gave only incorrect guesses when they used the Recalling Sums scheme.

We infer that children who had constructed the more sophisticated counting unit types, also constructed the higher level adding schemes. Thus Hendry, the counter of perceptual unit items used only the lowest scheme, Counting All correctly. Monica, the counter of motor unit items was able to use in addition to the Counting All scheme, Separating scheme to solve a missing-addend task. The counter of verbal unit items used the next higher scheme, Counting From 1. Cullen, Jeff, Paris, and Valerie, the counters of abstract unit items used the second highest level scheme, Counting On and also most of them correctly employed the Recalling Sums scheme. This is consistent with the finding by Carpenter (1983b) "that children initially solve (addition) problems with a Counting All strategy and that this strategy gradually gives way to Counting On and the use of number facts (Recalling Sums)" (p. 23). The findings in this study are also consistent with Houlihan and Ginsburg's (1981), that second graders use counting on procedures more than first graders. But Cullen and Valerie who were first graders used more advanced schemes than Monica a second grader. Also some of the older children have not constructed the most advanced schemes. For example, Monica was of the same age as Jeff but she used lower level schemes than Jeff and solved only a few more problems than Hendry, who was 18 months younger. Another significant finding was that Cullen, although of the same age as Hendry, used higher level schemes and solved nearly as many problems as did Jeff, who was 18 months older. Also, John who was younger than both Jeff and Monica used the highest level schemes, and solved nearly all problems by r calling number facrs. Another significant observation was that John, Jeff, and Monica were from the same second-grade classroom, just as Cullen and Hendry belonged to the same first-grade classroom.

## Adding Schemes and Problem Types

Table 10 shows that the children who used the most advanced adding schemes also solved all the three types of addition problems, direct addition and missing-addends (first or second). The problems solved also involved larger addends,

Insert Table 10 about here

some between 10 and 20 and others greater than 20. Each child usually used his or her most advanced scheme to solve problems involving smaller addends and the least advanced scheme (identified for that child) to solve problems with larger addends. This relation between schemes and problems was observed also within the specific schemes of the same representation. For example, Jeff used the Counting On-Using Abstract Unit Items scheme to solve problems with both addends less than 10 but used the Counting On-Using Motor Unit Items scheme to solve those with one addend greater than 20. This indicated that a child was more likely to use the most efficient scheme to solve a problem if she was capable of employing that scheme with confidence (minimum error). This finding compares favorably with the hypotheses of Briars and Larkin (1982) that when alternative strategies are available a child will respond with the strategy that results in the fewest counting procedures. But this hypothesis should be viewed as indicating the child's awareness of increased error in counting with larger addends. (cf. Shani's above remark that she started with 8 because "that is the hard one").

There was a noticeable difference between the missing-addend problems solved by the children who used the Counting On and the Recalling Sums schemes correctly and those who did not. The former solved many missing-addend problems including those with addends greater than 20 or between 10 and 20. On the other hand, the children who used the Counting From 1 and the Counting All schemes correctly, solved only a couple of missing-addend problems and none with addends greater than 10. The problems for the latter children had to be repeated a number of times and presented in stages to emphasize the actions involved before the children succeeded to solve the problems. Steffe et al. (1983) made similar observations about the performance of counters of perceptual, motor, and verbal unit items when presented with missing-addend problems. The difficulty experienced by the counters of sensory-motor items might be partly due to their intention to establish one collection when solving tasks. Thus they do not decide before counting to keep track of their counting acts beyond the first (given) addend. Their success generally depends on the interviewer intervening, after the child has established a collection for the given addend, and repeating the remaining part of the problem.

## Subtracting Schemes

Subtraction is the inverse of addition, and therefore, it is an operation that can be performed mentally (internalized action) or executed materially as an observable action scheme. The subtracting schemes that have been identified and which involved counting, that is, Separating, Adding All, Trial and Error Separating, Counting Up, and Counting Down schemes reflect observable, materially executed action schemes. On the other hand, the Recalling Number Facts and the Counting Up-Using Abstract Unit Items schemes identified reflect internalized actions for subtracting.

### Developmental Levels in Subtracting Schemes

Separating. This was the lowest level of the representations of children's subtraction concepts identified in the study. This scheme can be compared to Carpenter (1983b), and Carpenter and Moser's (1982) Separating From strategy. This was the only scheme that Hendry succeeded to construct, with guidance from the interviewer. His initial scheme for subtracting was to guess an answer and explain by counting fingers equal to that number. For example, to solve "5 - 2", Hendry said, "4" and explained by sequentially extending four fingers while uttering "1,2,3,4".

Five other children, Cullen, Monica, Paris, Shani, and Valerie also used the Separating scheme (see Table 11). Monica, Shani and Valerie used only this scheme to solve direct subtraction problems (e.g. 9 - 6 = ____). This finding was significant in the case of Valerie, who was a counter of abstract unit items.

Adding All. This scheme was identified as the next higher level after the Separating scheme. It was applicable only for solving missing-minuend problems. Monica was the only counter of sensory-motor items to use this scheme, but this was only after she had been guided by the interviewer to solve similar problems by, seeing the subtrahend and the difference (parts) as contained in the missing-minuend (whole). Cullen and Paris were the only counters of abstract unit items to use this scheme.

Trial and Error Separating. This scheme was identified as a higher level representation than the Separating and Adding All schemes. The scheme was used by Cullen and Jeff, both counters of abstract unit items. Even though the scheme appeared simple (from the adult's view), the child who used it indicated some awareness of the subtrahend and the difference as parts of the minuend (whole). For example, to solve "(__ - 5 = 4)" with this scheme, Jeff's intention was to construct a collection of eight items, separate five items (the subtrahend) and check if the remaining items will be four (the given difference). Thus when his first constructed collection of eight items failed to leave the expected four items, Jeff added 1 item, formed a new collection, and repeated the process.

Counting Up. This scheme was used on more occasions than the Counting Down scheme. Only the counters of abstract unit items used the scheme. This finding was consistent with the failure of the counters of sensory-motor items to use the Counting On scheme to solve addition problems. This scheme was, therefore, identified as higher than the Trial and Error Separating, Adding All, and Separating schemes. The Counting Up-Using Perceptual

Unit Items scheme can be compared to Carpenter and Moser's (1982) Adding On, with concrete objects strategy. The latter's Counting Up From Given strategy can be compared to the Counting Up-Using Verbal or Motor Unit Items scheme identified in the study.

Counting Down. We identified the Counting Down scheme at a higher level than the Counting Up scheme. The support for our hypothesis is, first, Valerie used the Counting Up scheme to solve a variety of problems (see Tables 12 and 13), but she never used the Counting Down scheme. Second, Cullen used the Counting Down scheme on only one occasion to solve "30 - 6". Third, none of the counters of sensory-motor items was able to count backward, even with perceptual items, to solve a task. This finding was consistent with that by Steffe et al. (1983) The latter explained the inability of counters of sensory-motor items to count backward as follows:

> The conceptual requirements for separating items from a collection by counting backward include the understanding that a number word that refers to a particular item of a collection also refers to those items yet to be counted (p. 102).

The conceptual requirement for the child to use the Counting Down scheme using motor or verbal unit items, includes the ability to use reversibility of counting. In other words, the child must be aware that counting backward from, say 7 to 1 will involve the same counting number words as when counting forward from 1 to 7. Carpenter and Moser (1982) found that about half of the first-graders in their study could not count backward a given number of steps. This finding supports the hypothesis that first-graders are likely to experience difficulty in using reversibility of counting. Our support for this hypothesis was provided by Cullen and Valerie, both first-graders, and the only counters of abstract unit items who hardly used the Counting Down scheme, besides John who used even higher level schemes. Carpenter and Moser (1982) also found that the Counting Down scheme was difficult for children to use.

Recalling Number Facts

We identified the Recalling Number Facts scheme as the highest level of the representation of subtraction. Hendry, Monica, and Shani, counters of sensory-motor items, failed to use any recall of number facts to solve subtraction problems. Their guesses were incorrect and were followed by more guesses. Paris, Valerie, Cullen, and Jeff used correctly at least two of the five Recalling Number Facts schemes (see Table 11). The latter three used the Recalling Addition Fact and Recalling Subtraction Fact

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Insert Table 11 about here

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schemes, in additiion to the Recalling Result By Guessing scheme. These children succeeded to use only the Recalling Sum By Guessing to solve addition problems. But their recall of facts were no better for subtracting than for adding. This was not surprising, since the Recalling Addition and Subtraction Fact schemes were indirect recall of number facts. For example, when Jeff solvedd "9 - ___ = 4" by recalling "5 and 4 makes 9", he had to recall his own number fact and relate it to the problem.

John used all the five Recalling Number Facts schemes. He also did not use any scheme that involved counting. Subtraction was completely an internalized activity for John, just as it was for addition.

The ability to use the Recalling Addition Fact to solve subtraction problems indicated that the child realized subtraction as the inversion of addition. However, with the exception of John, the other four counters of abstract unit items did not use the Recalling Addition Fact to solve direct subtraction problems (e.g. 18 - 7 = ___). Cullen and Jeff used the scheme to solve missing-subtrahend problems, while Cullen, Paris, and Valerie used the scheme to solve comparison-more problems (e.g. how many more is 5, the child's input than 3, the interviewer's input) (see Table 5(a)). Only Jeff used the scheme to solve equalizing-add problems (e.g. how many must the machine add to 5 to make 8). Also only Valerie and John used the Recalling Subtraction Fact scheme. We infer that direct subtraction problems trigger mainly counting in the four counters of abstract unit items, excluding John. In fact, as mentioned earlier, Valerie used only the Separating scheme to solve direct subtraction problems.

Steffe et al, (1983) have pointed out that a child has to be able to partition a number (minuend) into parts (subtrahend and difference), make a reversal of the parts if necessary, and coordinate this with reversibility of counting before realizing that subtraction is the inverse of addition. The subtraction problems Steffe et al. used to make the above inference were presented immediately after the related addition problems. For example, "19 - 12 = ___" was presented to Christopher immediately following "12 + 7 = ___" (see p. 108). We did not present related addition and subtraction problems immediately following one another. Thus the children had no recent counting activity to re-present to themselves and reverse to make an inversion to solve direct subtraction problems. They might have done so if subtraction and addition had been completely internalized, as it was for John. Because they could then re-present and solve mentally the related addition problems and check to see if the sum was equal to the minuend. For example, to solve "24 - 12", John recalled the addition fact "12 + 12 = 24", so he answered, "12". But John first tried "14 + 12" and realized that "14" could not be the answer, because the latter sum was 26 and not 24. The other four counters of abstract unit items would need to perform counting activities to find "14 + 12" and "12 + 12" before they could use their knowledge of inversion to answer the original problem, "24 - 12". But there was no need for these four children to perfom two counting activities, when they could perform one counting activity to obtain the same result. We infer that these four counters of abstract unit items did not face the same situation with the other subtraction problems, hence their success in using the Recalling Addition Fact scheme. Our hypothesis is consistent with the inability of the counters of sensory-motor items to use the Recalling Addition Fact scheme to solve even the non-direct subtraction problems. Because the latter lacked the ability to use the Recalling Sum by Guessing scheme correctly to solve addition problems.

Subtracting Schemes and Problem Types

Hendry, Monica, and Shani, who were counters of perceptual, motor, and verbal unit items respectively, used only the Separating scheme to solve all direct subtraction problems. Hendry used only blocks to solve all his problems, which were also only direct subtraction. He, initially, used no correct scheme for subtracting until the interviewer guided him to separate a collection of items representing the smaller number (subtrahend) from the collection of items representing the larger number (minuend) to solve some problems. Only two of the problems presented to Hendry involved numerals on card as inputs, the rest were presented with blocks in sandwich bags as inputs. Hendry could not use his fingers to solve any problems in which the minued was greater than five.

Monica used the Separating scheme to solve also missing-subtrahend and equalizing problems. She succeeded to use the Adding All scheme to solve missing-minuend problems only after some initial guidance from the interviewer. But she showed no understanding of the comparison-more problem, "how many more is five than three"? She said, "five is lesser than three", and failed to solve the problem, despite the fact that she was made to look at her five extended fingers and three extended fingers of the interviewer. She, however, understood and solved equalizing-take away and equalizing-add problems involving numbers less than 10 (see Tables 12 and 13).

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Insert Tables 12 and 13

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Shani used the Separating scheme to solve also the missing-subtrahend and missing-minuend problems. She used her fingers more than blocks to represent numbers in the problems. Unfortunately her tape while solving comparison and equalizing problems could not be used due to malfunctioning of the camera. She also missed school on two days make up days.

We infer that the counters of sensory motor items used only one scheme, Separating, to solve all types of subtraction problems. The use of the Adding All scheme by Monica was triggered, when she was guided to see that the subtrahend and difference combine to make the minuend. But this scheme is consistent with her scheme for adding, that is, Counting All.

There was no simple relationship between the schemes and problem types for the counters of abstract unit items. However, individual children tended to use particular schemes for some problem types. For example, Valerie used only the Separating scheme and Recalling Result By Guessing schemes to solve all direct subtraction problems. But these children tended to use the higher level schemes when the possiblity of error was minimal, that is, the difference was less than 10. For example, John used the Recalling Addition Fact scheme to solve "9 - ____ = 4", but used the Trial and Error addition to solve "24 - 12" and "48 - 24". Further support for our hypothesis was given by both Jeff and Paris who failed to use a higher level scheme to solve problems with larger differences. For example, Jeff solved correctly "13 - ____ = 9" by using the Counting Down To- Using Verbal Unit Items scheme, but failed to solve the very next problem "34 - ____ = 20" with the same scheme. Similarly, Cullen and Valerie used the

Recalling Addition and Subtraction Fact schemes to solve missing-subtrahend problems when the minuend was less than 10, but they used the Separating scheme for larger minuends.

Cullen, Jeff, Paris, and Valerie initially found the missing-minuend problems difficult, so they used lower level schemes, but later used higher level schemes. For example, Jeff, initially, used the Trial and Error Separating scheme to solve "__ - 5 = 4", but later, he was able to solve "____ - 7 = 11" by using the Counting Up-Using Verbal Unit Items scheme.

All the five counters of abstract unit items had no difficulty in understanding and solving comparison and equalizing problems. The children's schemes for solving these problems were consistent with their schemes for adding and subtracting. For example, Valerie used the Counting Up and Recalling Addition Fact to solve comparison-more and equalizing-add problems, but she used the Separating scheme to solve the equalizing-take away problems. No child used the Counting Down scheme to solve comparison problems. Jeff's use of the Counting Down scheme to solve equalizing-take away problems was consistent with his use of the scheme for subtracting.

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Table 1

Addition problems presented with the function machine

| Input Numeral | | Output Numeral |
|---|---|---|
| First Addend | Second Addend | Sum |
| 4 | 2. | – |
| 2 | 5 | – |
| 5 | 2 | – |
| 4 | 3 | – |
| 3 | 4 | – |
| 5# | 3# | – |
| 8 | 3 | – |
| 3 | 8 | – |
| 5* | 6* | – |
| 4 | 9 | – |
| 4 | 10 | – |
| 10 | 4 | – |
| 6* | 8* | – |
| 8 | 7 | – |
| 4 | 11 | – |
| 13 | 15 | – |
| 23 | 10 | – |
| 30 | 13 | – |
| 34 | 10 | – |
| 23 | 30 | – |
| 45 | 16 | – |
| 16 | 45 | – |

* Objects in a bag # Pictures on a card

Table 2

Missing-Addend problems presented with the function machine

| Input Numeral | | Output Numeral |
|---|---|---|
| First Addend | Second Addend | Sum |
| 2* | – | 6* |
| 2 | – | 7 |
| 5 | – | 7 |
| 5 | – | 8 |
| 8 | – | 11 |
| 10 | – | 14 |
| 13 | – | 25 |
| 15 | – | 31 |
| 30 | – | 44 |
| 10 | – | 44 |
| 34 | – | 44 |
| – | 9 | 13 |
| – | 11 | 15 |
| – | 10 | 24 |
| – | 10 | 44 |

* Objects in a bag

Table 3

Subtraction Problems Presented with the Function Machine

| Input Numeral | | Output Numeral |
|---|---|---|
| Minuend | Subtrahend | Difference |
| 5 | 3 | - |
| 6 | 5x | - |
| 6 | 4 | - |
| 6 | 2x | - |
| 6 | 2* | - |
| 7 | 3 | - |
| 8 | 4 | - |
| 8 | 3x | - |
| 8 | 3 | - |
| 8 | 2 | - |
| 9 | 7 | - |
| 9 | 6 | - |
| 9 | 5 | - |
| 9 | 3 | - |
| 13 | 4 | - |
| 14 | 7 | - |
| 18 | 11 | - |
| 18 | 7 | - |
| 24 | 12 | - |
| 30 | 6 | - |
| 35 | 10 | - |
| 48 | 24 | - |

Legend:
- \- child to figure out
- x blocks in sandwich bags
- * pictures of animals on cards

Table 4

Missing - Subtrahend and Minuend Problems Presented with the Function Machine

| Input Numeral | | Output Numeral |
|---|---|---|
| Minuend | Subtrahend | Difference |
| [illegible] | [illegible] | 3* |
| [illegible] | [illegible] | 3 |
| 5 | [illegible] | 2 |
| 6 | [illegible] | 3 |
| 8 | [illegible] | 5 |
| 9 | [illegible] | 5 |
| [illegible] | [illegible] | 1 |
| [illegible] | [illegible] | 0 |
| 14 | [illegible] | 7 |
| 18 | [illegible] | 11 |
| 23 | [illegible] | 19 |
| 30 | [illegible] | 24 |
| 31 | [illegible] | 20 |
| [illegible] | 2 | 1 |
| [illegible] | 2 | 3 |
| [illegible] | [illegible] | 4x |
| [illegible] | 3 | 5* |
| [illegible] | 3 | 5 |
| [illegible] | 3 | 4 |
| [illegible] | 6 | 3 |
| [illegible] | [illegible] | 7 |
| [illegible] | 8 | 3 |
| [illegible] | [illegible] | 11 |
| [illegible] | 13 | 12 |
| [illegible] | 16 | 24 |

[illegible] - child to figure out
[illegible] blocks in sandwich bags
[illegible] picture of animals

Table 5(a)

Comparison - More Problems presented with the Function Machine

| Input Numeral | | Output Numeral |
|---|---|---|
| Child's | Interviewer's | |
| 4 | 2 | - |
| 5 | 3 | - |
| [illegible] | 9 | - |
| 1[illegible] | 14 | - |
| 11 | 7 | - |
| 15 | 9 | - |
| 18 | 6 | - |
| 12 | 25 | - |
| 25 | 13 | - |

Tabl 5(b)

Comparison - Less P[illegible]ms Presented with the Funct[illegible]ine

| Input Numeral | | Output Num[illegible] |
|---|---|---|
| Child's | Interviewer's | |
| [illegible] | 11 | - |
| 15 | 4 | - |
| 18 | [illegible] | - |
| 18 | 11 | - |
| 18 | 13 | - |
| 19 | 23 | - |

Legend: - Child to figure out

Table 6(a)

Equalizing - Add Problems Presented with the Function Machine

| Input Numeral | | Output Numeral |
|---|---|---|
| Child's | Interviewer's | |
| 5 | 8 | - |
| 7 | 6 | - |
| [illegible] | 14 | - |
| 12 | 25 | - |
| 23 | 15 | - |

Table 6(b)

Equalizing - Take Away Problems Presented with the Function Machine

| Input Numeral | | Output Numeral |
|---|---|---|
| Child's | Interviewer's | |
| 3 | 5 | - |
| 13 | 9 | - |
| 18 | 11 | - |
| 7 | 18 | - |
| 30 | 24 | - |
| 61 | 45 | - |

Legend: - child to figure out

Table 7

Children's Description of Addition by a Function Machine

| Name of Child | Description of Addition |
| --- | --- |
| John, Jeff | Add |
| Cullen | Adding |
| Valerie, Paris | Put ... together to make |
| Monica, Shani | Take away ... put them on here |
| Hendry | Make more numbers |

Table 8

Children's Description of Subtraction by a Function Machine

| Name of Child | Description of Subtraction |
|---|---|
| John | Subtract |
| Cullen, Jeff | Take away |
| Paris, Valerie | Take away |
| Shani | Changed it from 6 to 4 |
| Monica | It took (remove) two away |
| Hendry | Make less |

Table 9. Children's Adding Schemes

| Scheme | | John | Jeff | Cullen | Valerie | Paris | Shani | Monica | Hendry |
|---|---|---|---|---|---|---|---|---|---|
| Recalling Sums | Using Place Value | / | | | | | | | |
| | To A Decade | / | | | | | | | |
| | Using Doubles | / | | | | | | | |
| | By Guessing | / | / | / | / | | | | |
| Adding On By Tens And Ones | | / | | | | | | | |
| Counting On | Using Abstract u/i | | / | | / | | | | |
| | Using Verbal u/i | | / | / | / | / | | | |
| | Using Motor u/i | | / | / | / | / | | | |
| | Using Perceptual u/i | | / | | / | | | | |
| Counting From 1 | Using Verbal u/i | | | | | | / | | |
| | Using Motor u/i | | | | | / | / | | |
| | Using Perceptual u/i | | | | | | / | / | / |
| Counting All | In Separated Collections | | | | | / | | / | |
| | In Joined Collections | | | | | | | | / |

Legend: / scheme used by child

u/i unit items

88

89

Table 10. Adding Schemes and Addition Problems

| SCHEME | | Direct Addition a + b = ___ | Missing-Addend a + ___ = c | Missing-Addend ___ + b = c |
|---|---|---|---|---|
| Recalling Sums | Using Place Value | Jo* | | |
| | To A Decade | Joα | | |
| | Using Doubles | Jo* | | |
| | By Guessing | Jo C V | Jo Je | Jo Je C V |
| Adding On By Tens And Ones | | | Joα | |
| Counting On | Using Abstract u/i | C V Je | Je | |
| | Using Verbal u/i | C V Jeα P* | Je* V | Je |
| | Using Motor u/i | Cα V* Jeα P* | Vα Cα P* | C* V |
| | Using Perceptual u/i | Vα | Je* | |
| Counting From 1 | Using Verbal u/i | S | S | |
| | Using Motor u/i | P S | P | P |
| | Using Perceptual u/i | S | | |
| Counting All | In Separated Collections | Pα M | | |
| | In Joined Collections | H | | |
| Separating | | | M | Cα |

Legend: C-Cullen H-Hendry Je-Jeff Jo-John
M-Monica P-Paris S-Shani V-Valerie
u/i unit items
* one addend is between 10 and 20
α one addend is greater than 20

90

91

Table 11. Children's Subtracting Schemes

| SCHEME | | John | Jeff | Cullen | Valarie | [illegible]ris | Shan[illegible] | So[illegible]a | Hendry |
|---|---|---|---|---|---|---|---|---|---|
| | | Name of Child | | | | | | | |
| Recalling | Derived Facts | / | | | | | | | |
| | Number Fact by Trial & Error | / | | | | | | | |
| | Subtraction Fact | / | | / | / | | | | |
| | Addition Fact | / | / | / | / | / | | | |
| | Result by Guessing | / | | / | / | / | | | |
| Count Down To | Using Verbal u/i | | | | | | | | |
| | Using Motor u/i | | | | | / | | | |
| | Using Perceptual u/i | | | | | / | | | |
| Counting Down With | Verbal u/i | | / | | | | | | |
| | Motor u/i | | | / | | / | | | |
| | Perceptual u/i | | | | | / | | | |
| Counting Up | Using Abstract u/i | | / | | | | | | |
| | Using Verbal u/i | | / | | / | | | | |
| | Using Motor u/i | | / | / | / | | | | |
| | Using Perceptual u/i | | | | | / | | | |
| Trial and Error Separating | | | / | / | | | | | |
| Adding All | | | | / | | / | | / | |
| Separating | | | | / | / | / | / | / | / |

Legend: u/i unit items
/ scheme used by child

93

92

Table 12: Children's Schemes and Subtraction Problems

| SCHEME | | Direct Subtraction a - b = ___ | Missing-Subtrahend a - ___ = c | Missing-Minuend ___ - b = c |
|---|---|---|---|---|
| Recalling | Derived Fact | Jo | Jo* | |
| | Number Fact By Trial And Error | Jo* | Jo | |
| | Subtraction | | V | Jo |
| | Addition Fact | Jo^α | Jo Je C | |
| | Result By Guessing | Jo Je P | | |
| Count Down To | Using Verbal u/i | | Je* | |
| | Using Motor u/i | | Je^α P* | |
| | Using Perceptual u/i | P* | | |
| Count Down With | Verbal u/i | Je^α | | |
| | Motor u/i | P* C^α | | |
| | Perceptual u/i | P | | |
| Counting Up | Using Abstract u/i | | | Je |
| | Using Verbal u/i | Je | | Je V |
| | Using Motor u/i | C | | |
| | Using Perceptual u/i | | | |
| Trial And Error Counting | | | C* | Je C* |
| Adding All | | | | C P M |
| Separating | | P* C* V* S* M* H | V* C^α P S M | S |

Legend: C-Collen H-Hendry Je-Jeff Jo-John
M-Monica P-Paris S-Shani V-Valerie

u/i unit items

* minuend is between 10 and 20

α minuend is greater than 20

Table 13. Children's Schemes, and Comparison and Equalizing Problems

| SCHEME | | Comparison more | Comparison less | Equalizing-add | Equalizing-take away |
|---|---|---|---|---|---|
| Recalling | Derived Fact | | Jo | | ∝ Jo |
| | Number Fact By Trial and Error | ∝ Jo | | | |
| | Subtraction | | | | |
| | Addition Fact | V *P C | | Je | |
| | Result By Guessing | *Jo V | *Jo C | | |
| Count Down To | Using Verbal u/i | | | | |
| | Using Motor u/i | | | | *Je |
| | Using Perceptual u/i | | | | |
| Counting Up | Using Abstract u/i | *Je | | | |
| | Using Verbal u/i | | V | | |
| | Using Motor u/i | ∝V *Je | *Je | *Je V | |
| | Using Perceptual u/i | ∝P | P | | |
| Separating | | | V | P M | V P M |

Legend: C-Cullen Je-Jeff Jo-John
M-Monica P-Paris Valerie
* larger gien number is between 10 and 20
∝ larger given number is greater than 20

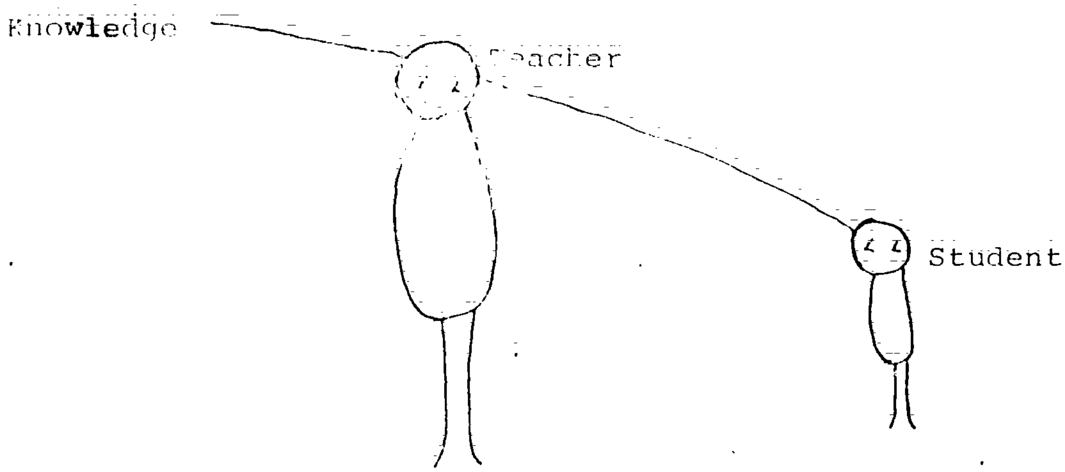



Figure 1: Subordination of Learning to Teaching

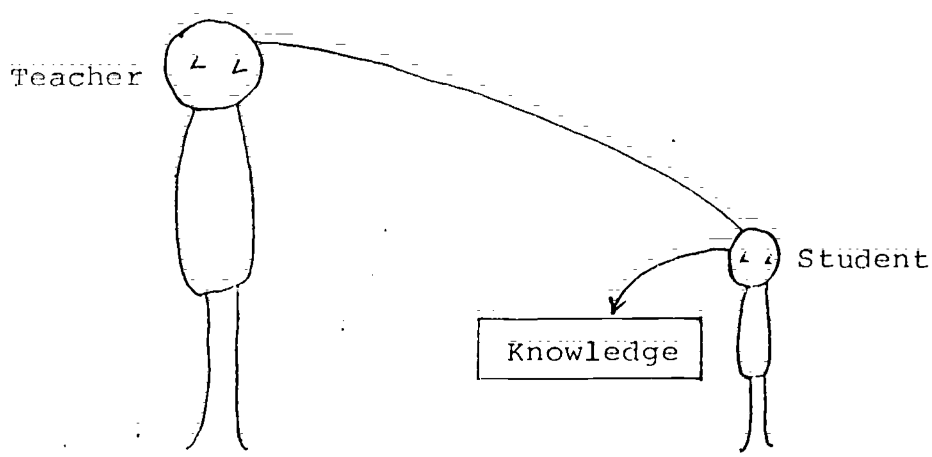



Figure 2. Subordination of Teaching to Learning

| Counter of | Perceptual items | Figural items | Motoric items | Verbal items | Abstract items |
|---|---|---|---|---|---|
| Abstract u/items* | / | / | / | / | o |
| Verbal u/items | / | / | / | o | |
| Motor u/items | / | / | o | x | |
| Figural u/items | / | o | x | x | |
| Perceptual u/items | o | x | x | x | |

*u/items: unit items

Legend: o most advanced items

/ more primitive items

x undifferentiated items

Figure 3. The development of counting

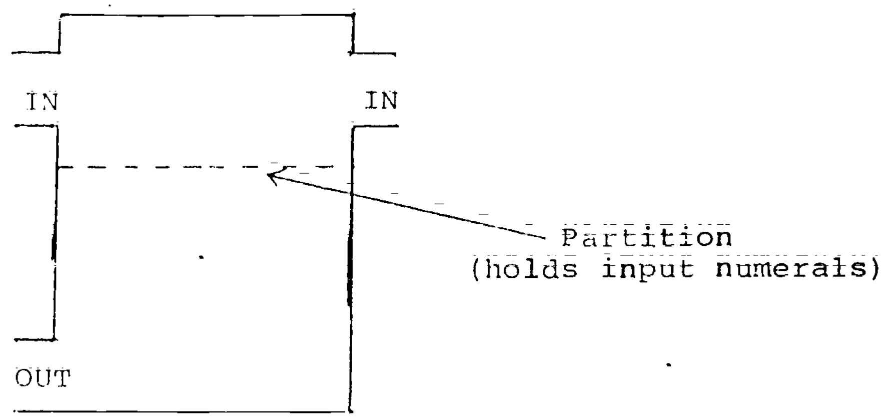



Figure 4(a). Front view of a Function Machine with two Input Holes

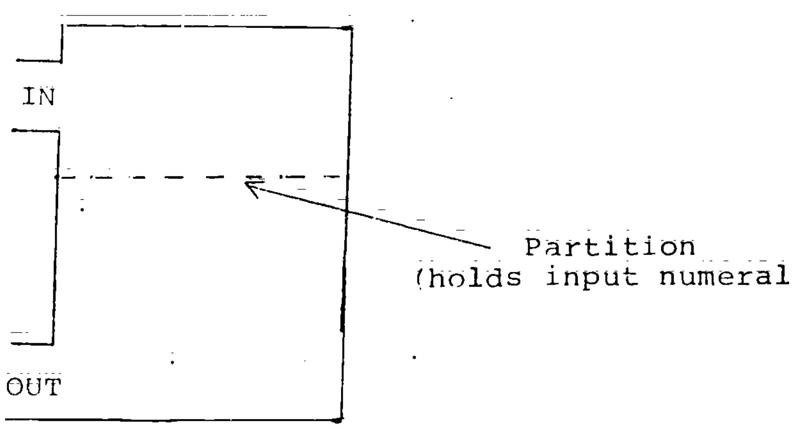
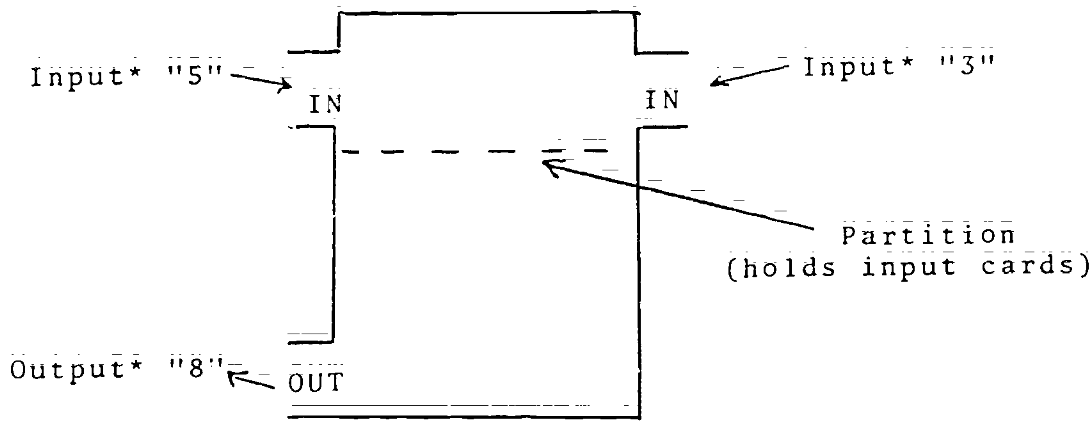
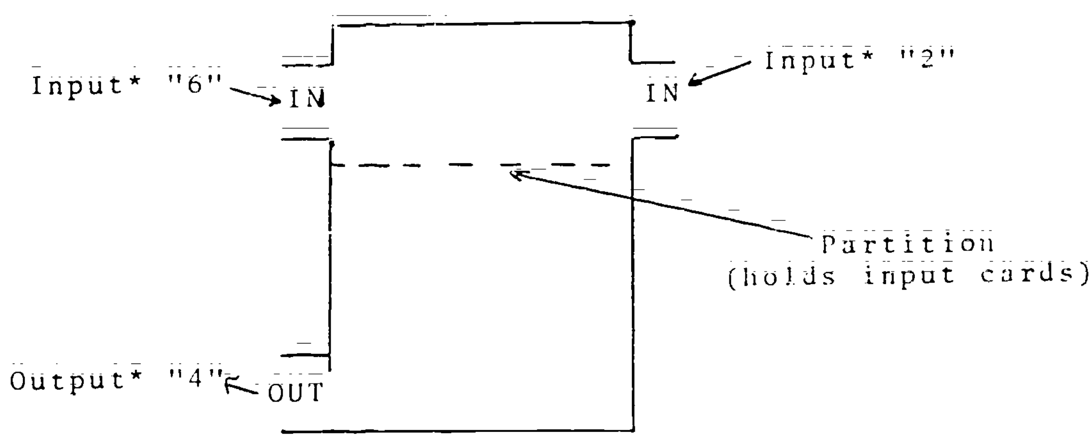



Figure 4(b). Front view of a Function Machine with one Input Hole




* card with pictures of animals

Figure 5(a). Addition by a Function Machine




* card with pictures of animals

Figure 5(b). Subtraction by a Function Machine

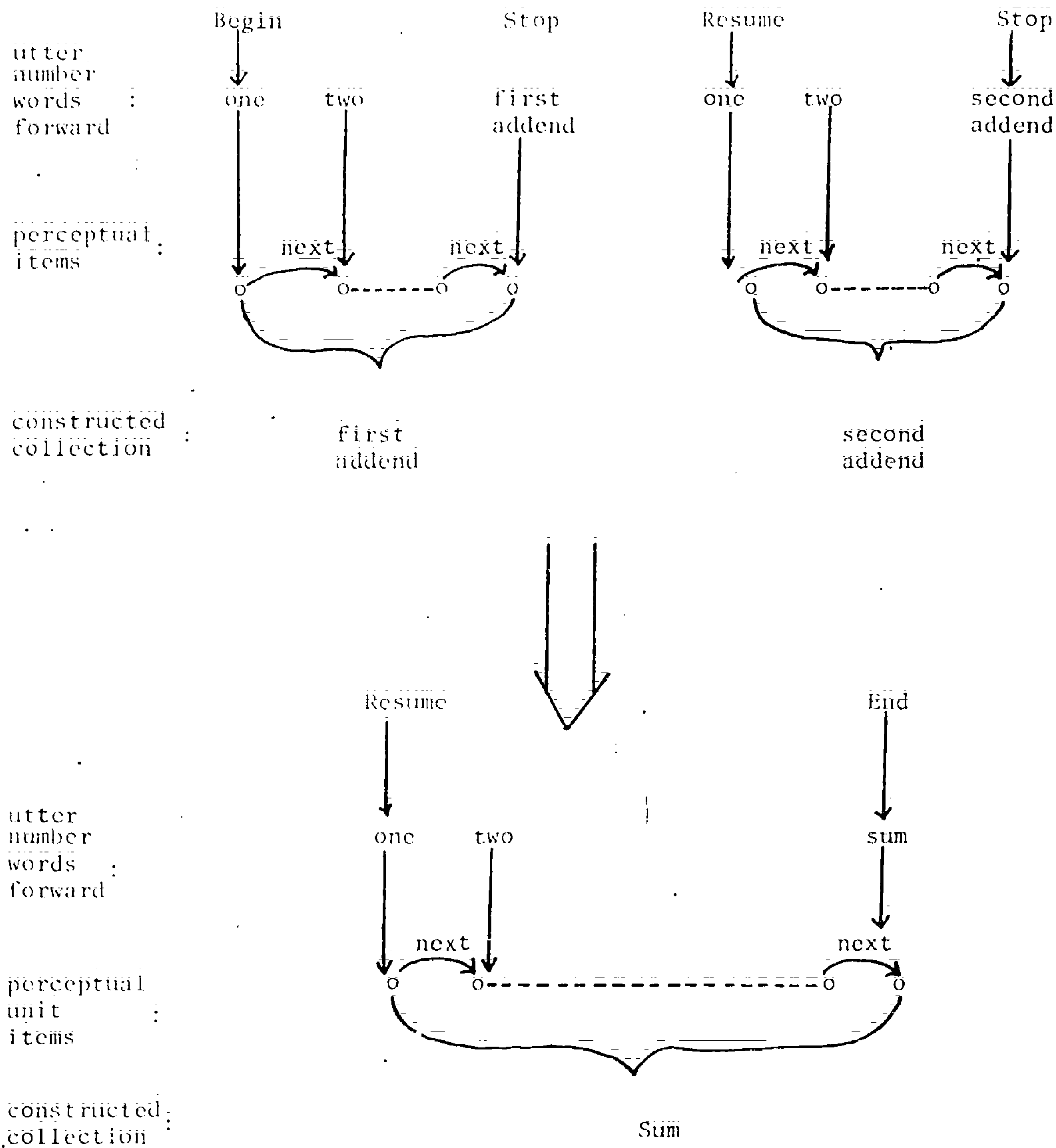



Figure 6(a) Representation A1 (Counting All In Joined Collections)

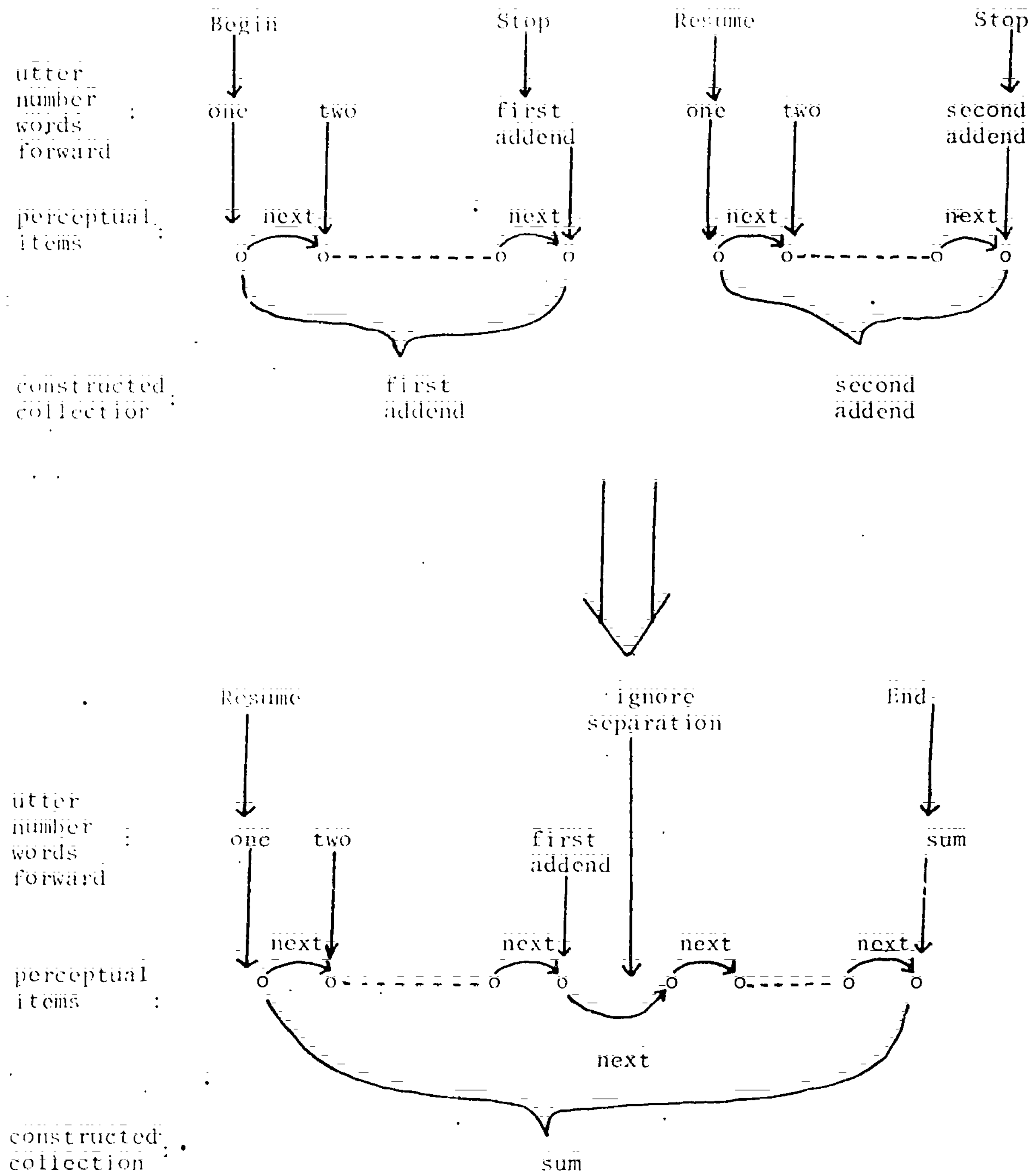



Figure 6(b) Representation A1 (Counting All In Separated Collections)

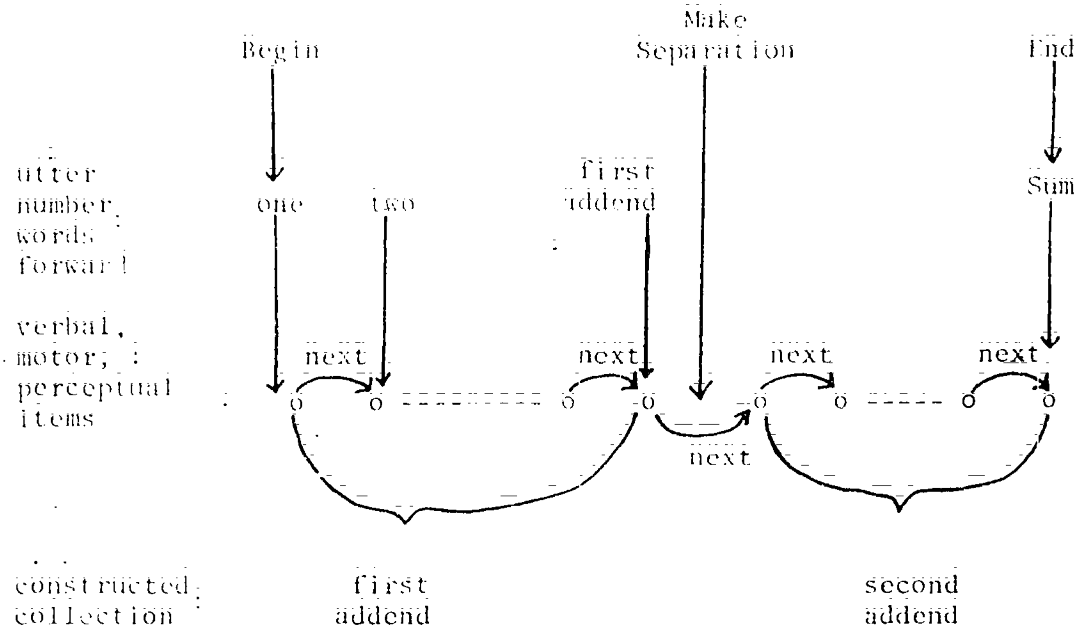



Figure 7. Representation A2 (Counting From 1)

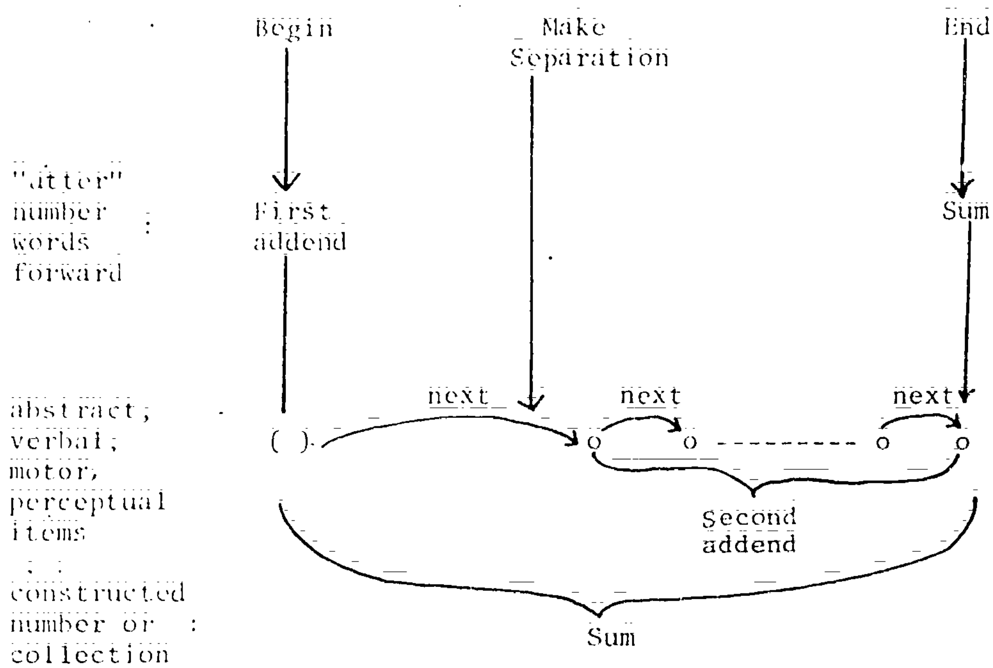



Figure 8. Representation A3 (Counting On).

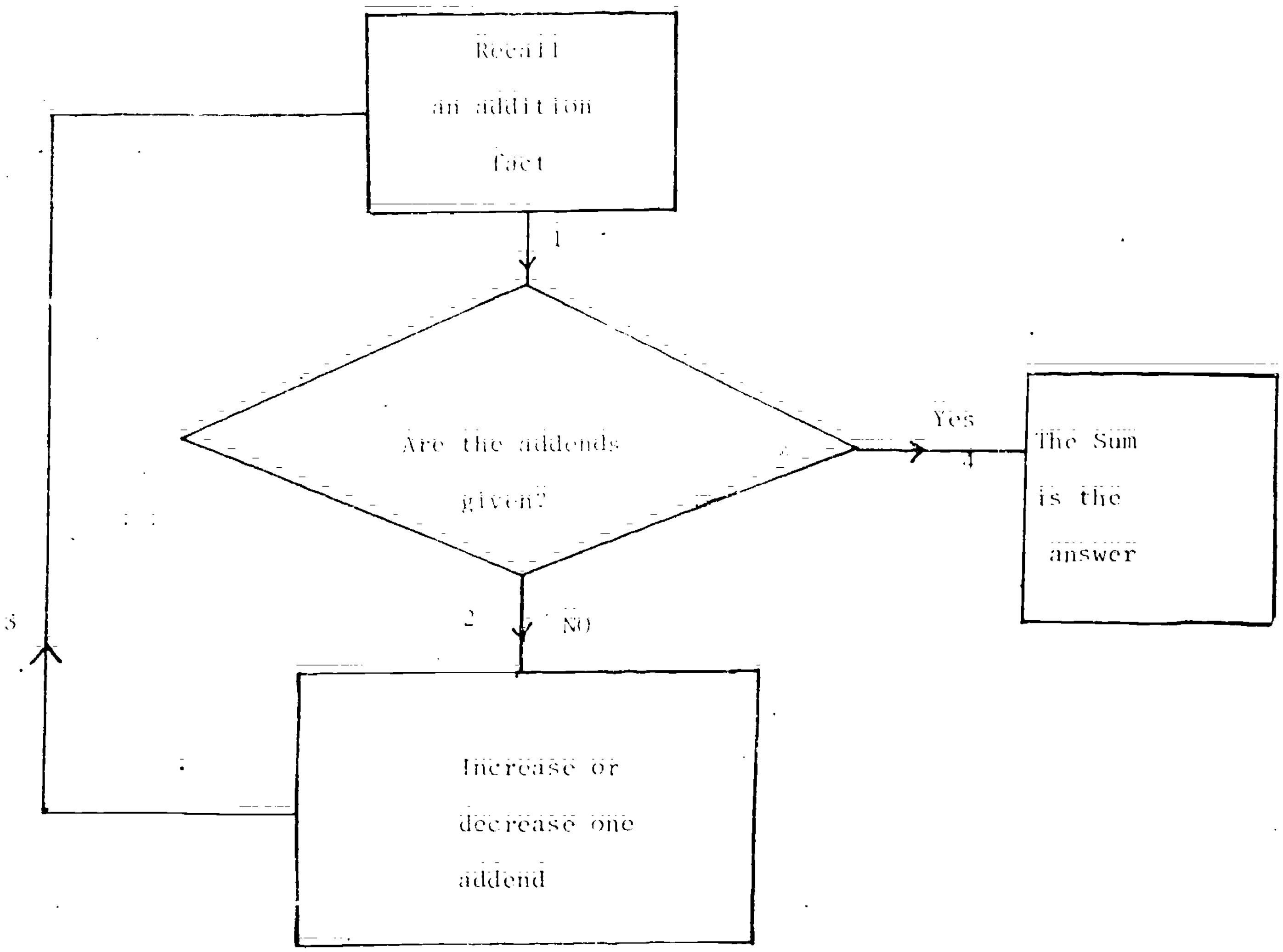



Figure 9. Representation A4 (Recalling Sums).

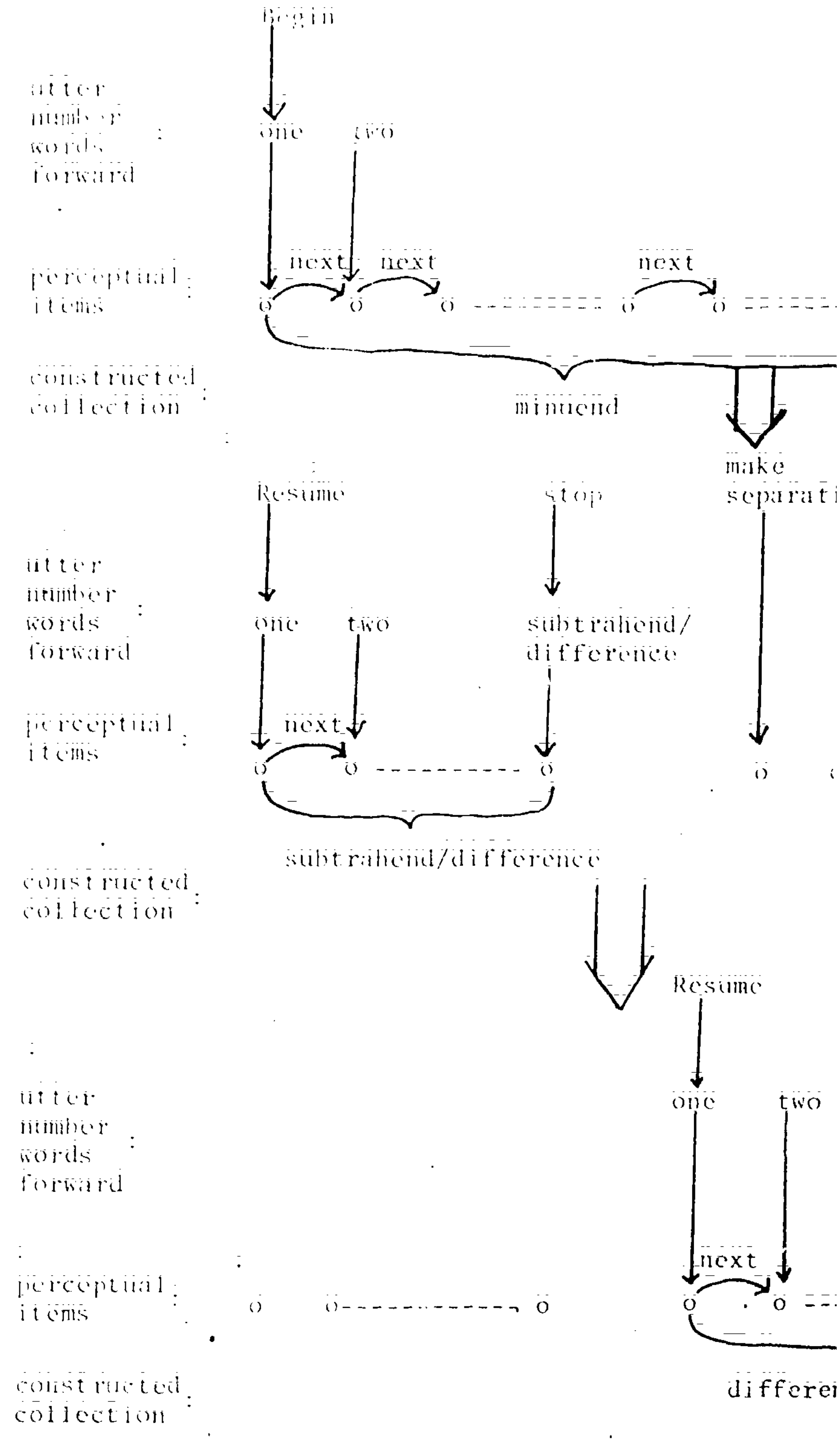



Figure 10. Representation S1 (Separating)

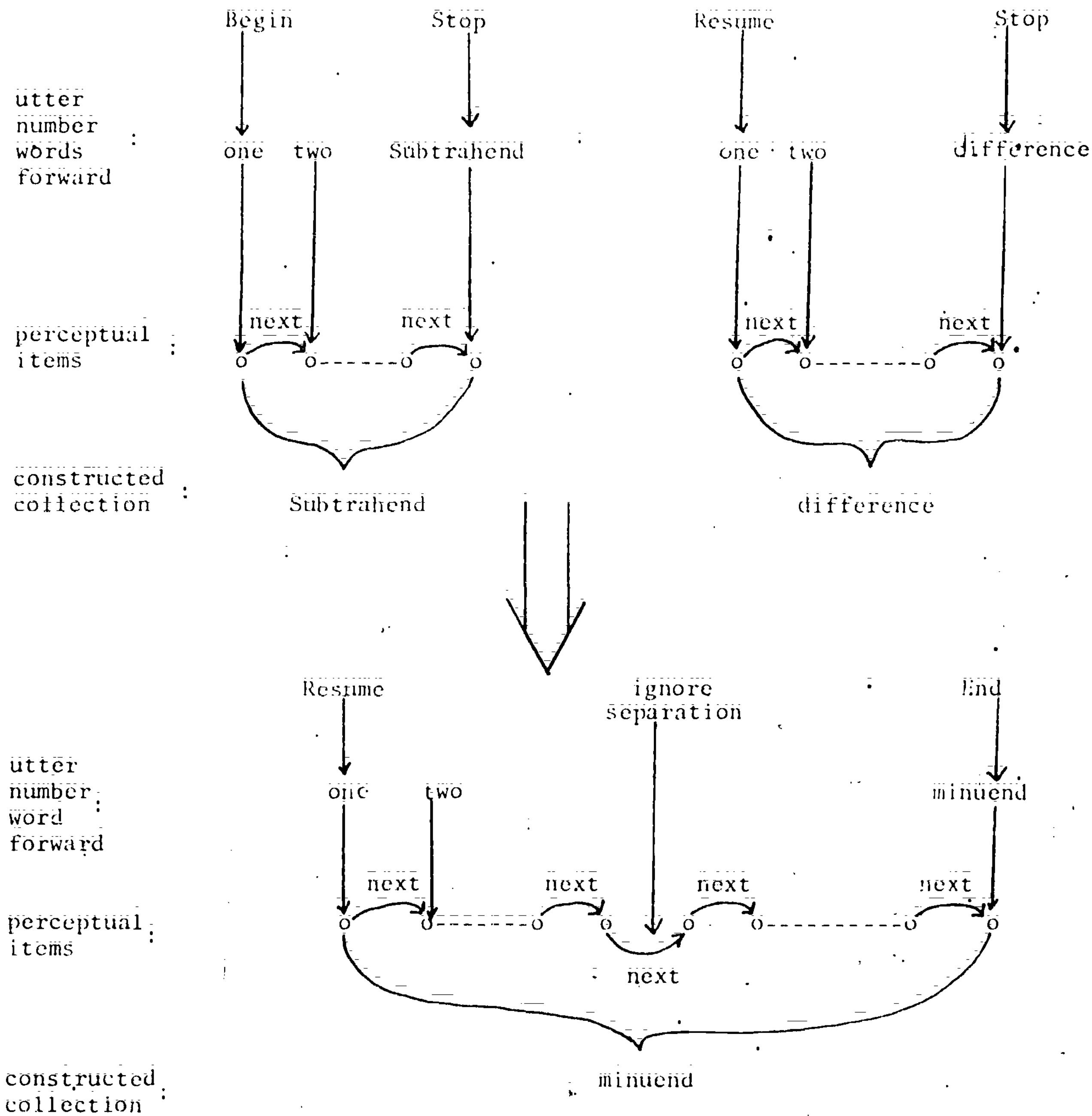



Figure 11. Representation S2 (Adding All)

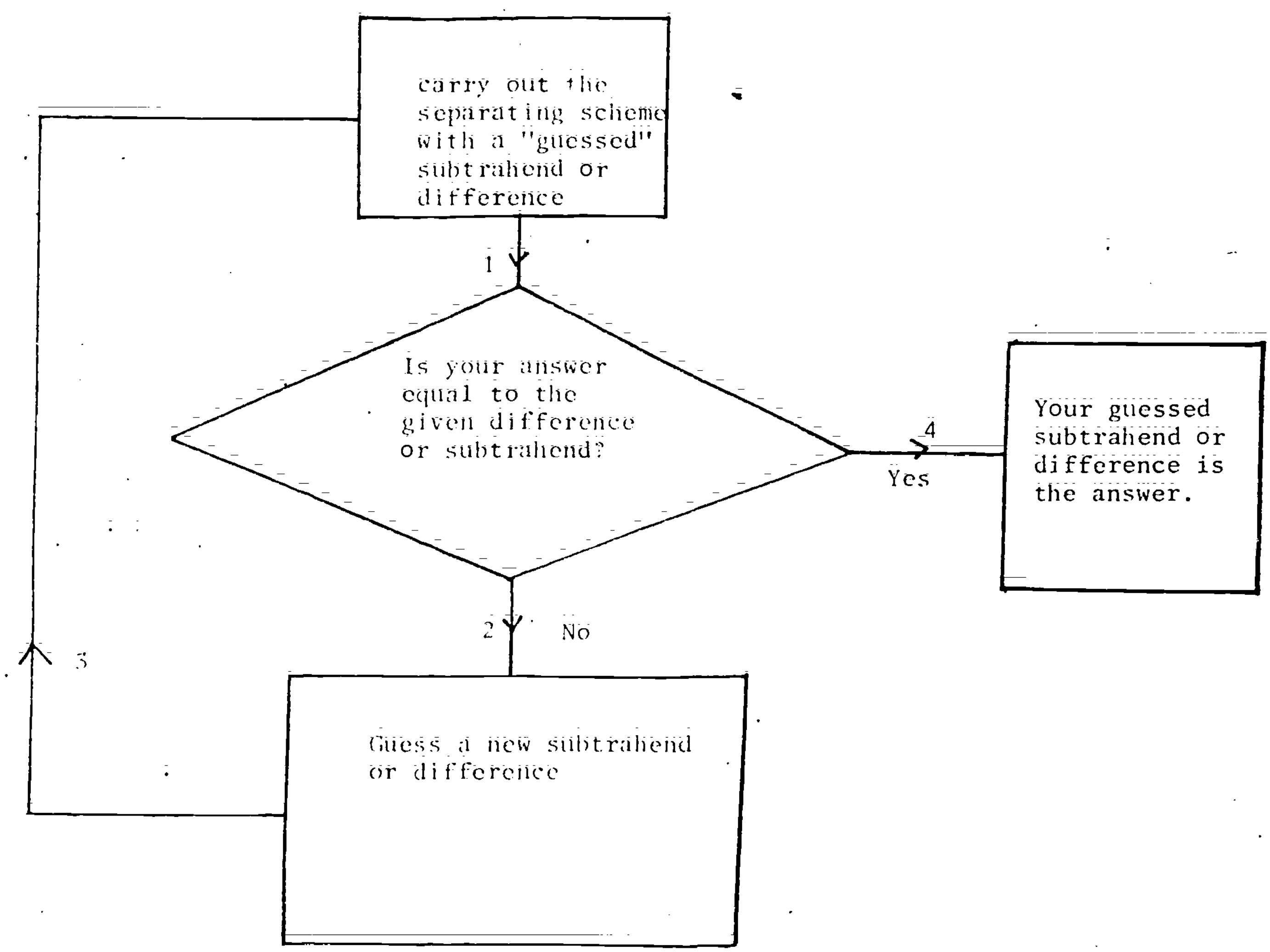



Figure 12: Representation S3 (Trial and Error Separating)

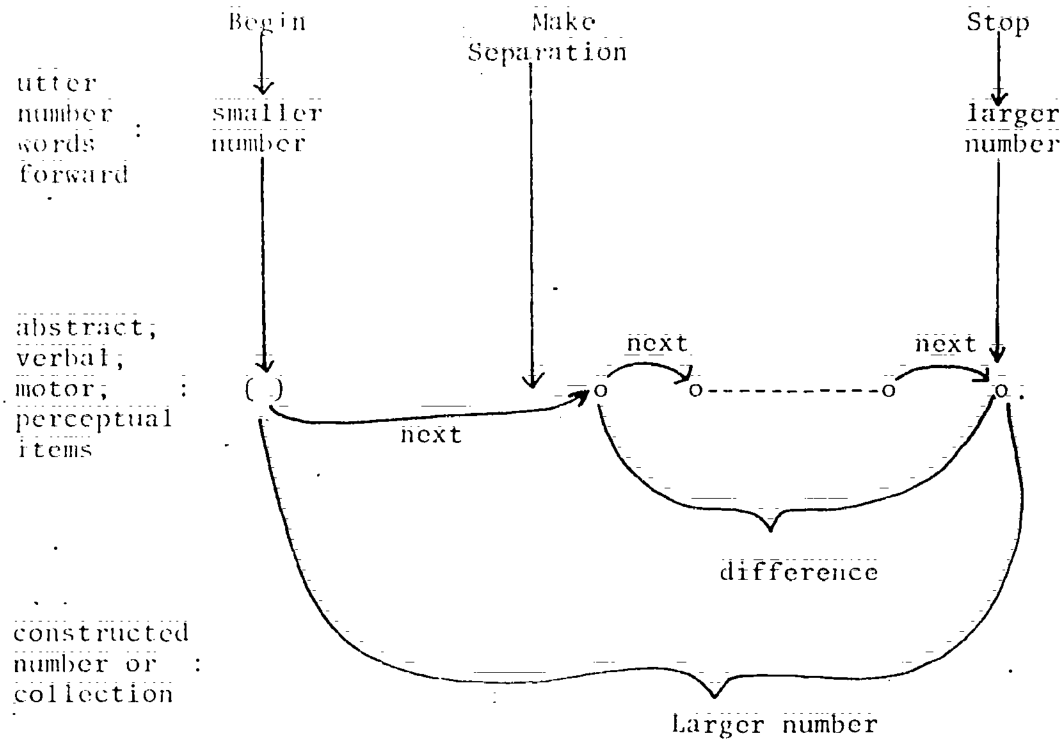



Figure 13: Representation S4 (Counting Up)

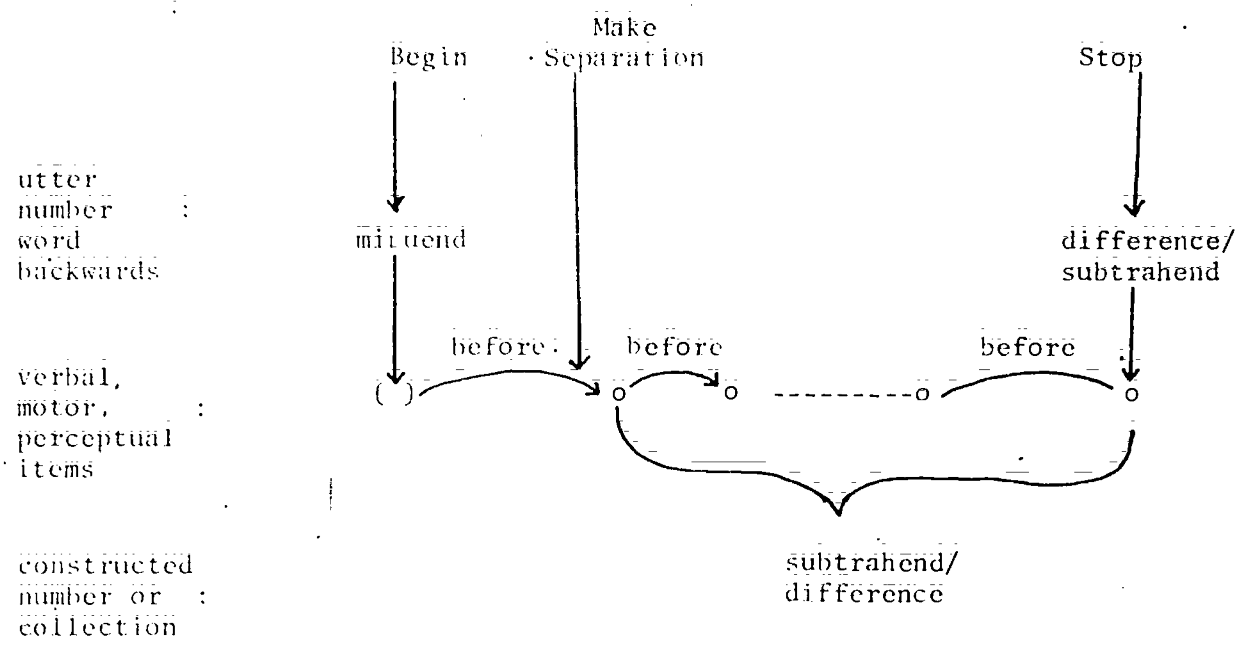



Figure 14. Representation S5 (Counting Down)

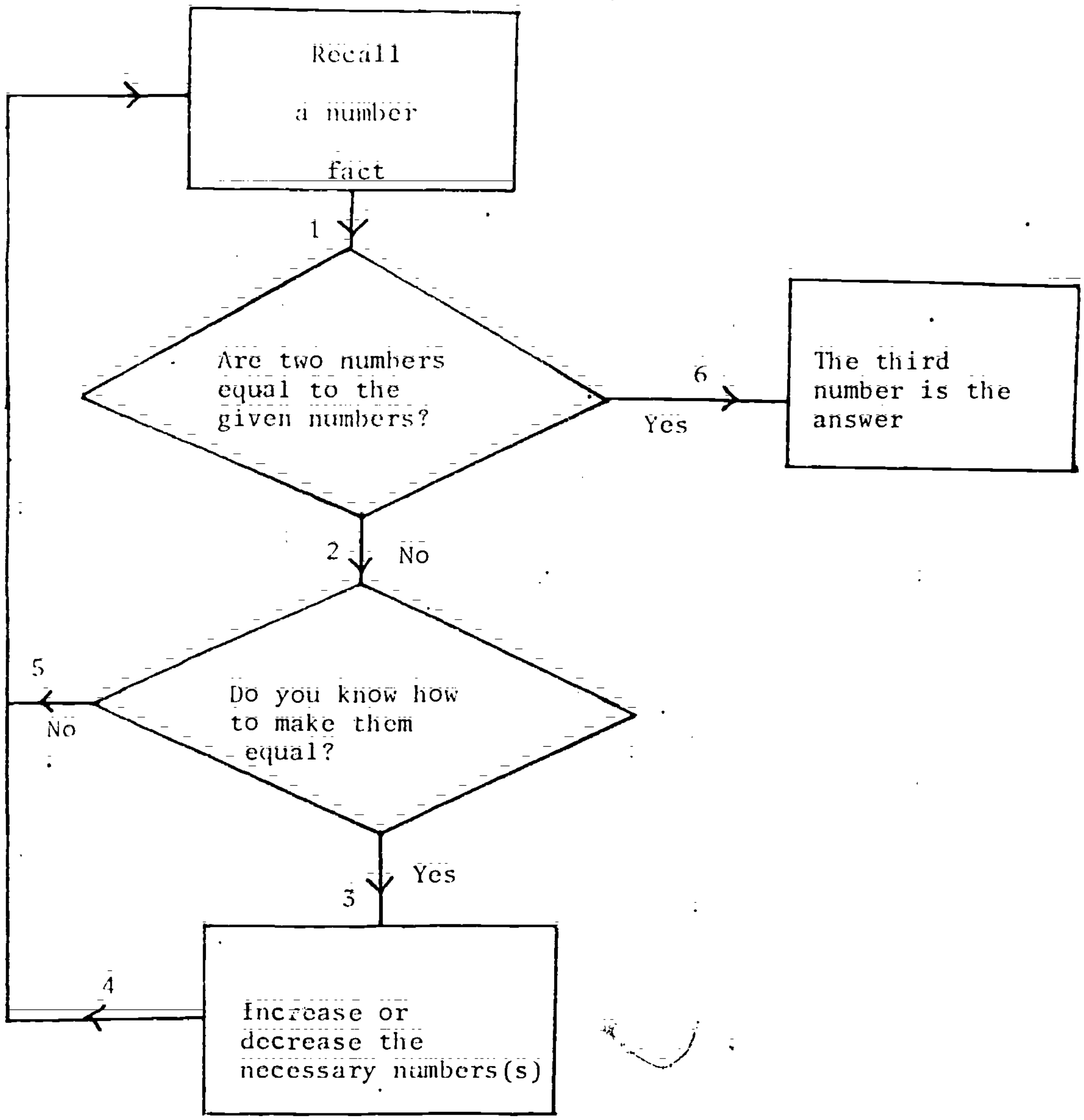



Figure 15. Representation S6 (Recalling Number Facts)